# On the Twistor Theory of Symplectic Manifolds

Rui Pedro L. P. Ribeiro de Albuquerque

A thesis submitted in partial fulfilment of the requirements of the University of Warwick for the degree of Doctor of Philosophy in Mathematics

Mathematics Institute, University of Warwick, Coventry, England

June 2002

# Acknowledgments

It is a great honour for me to write a few words in immense gratitude towards all those who helped me to have a wonderful time with higher mathematics.

Meaning the last four years, I wish to express my biggest thanks to my research supervisor, Professor John Rawnsley, for accepting me as a student and giving me such a wonderful theme of research. I must say the best times I had in beautiful England were passed in mathematical discussions with him. I got the most generous answer to any question I had, while many other which he adressed to me received later the same treatment from him. Hence the great contribution of J. Rawnsley to many results of this thesis. Moreover I wish to thank him the perspectives he gave me of Mathematics.

I am grateful also to all the staff in the Mathematics Institute for creating such an enthusiastic environment for Research.

Now I would like to thank Professors Armando Machado, Isabel Salavessa, José Ribeiro and Teresa M. Fernandes. Without them, nothing would had been possible: from graduation to the 'geometry school' of Lisbon, the perfect choice of Warwick and finally to Évora's big patience.

I am grateful to my comrades from PCP for giving me so many ideas to dream of, and for their respect for Work and love for Truth — an understanding of Truth that, I am convinced, much agrees with the scientific needs. It is to *that* struggle for Progress that I am surely indebted. Furthermore, the Portuguese Left owes to its most coherent force much more than she cares to admit.

My warmest thanks go to my big 'family': J. Ribeiro de Albuquerque, Eduardo Pinto, ..., Anabela Barata, Maria João Guia, ..., Marta Barata, José and João. They would have constitute motive enough for me to strive through Mathematics.

To Marta, to whom I just can say that I love, I am heavily indebted. My friends and colleagues from Anchor House, Évora, Lisbon and Warwick — my thanks go to them too.

I acknowledge the help of University of Évora and a scholarship from FCT, Praxis XXI, BD/11514.

For all special reasons, I dedicate this thesis to my parents: Maria Teresa and Rui Manuel.

Thou mak'st me see it wi' a better eye. Bless thee. Good night. Good-bye!"

It was but a hurried parting in a common street, yet it was a sacred remembrance to these two common people. Utilitarian economists, skeletons of schoolmasters, Commissioners of Fact, genteel and used-up infidels, gabblers of many little dog's-eared creeds, the poor you will have always with you. Cultivate in them, while there is yet time, the utmost graces of the fancies and affections, to adorn their lives so much in need of ornament; or, in the day of your triumph, when romance is utterly driven out of their souls, and they and a bare existence stand face to face, Reality will take a wolfish turn, and make an end of you.

a Thought to the world
from Charles Dickens

# Declaration

All material contained in this thesis is original work of the author, except that which was taken from the sources cited in the text.

# Summary

This thesis is a contribution to the understanding of both the real and complex geometry of the twistor space of a symplectic manifold. The latter is the natural bundle of all compatible complex structures and is defined over any manifold possessing a non-degenerate 2-form. Its fibre is the Siegel domain $Sp(2n,\mathbb{R})/U(n)$. Recall that, according to a choice of a linear connection on the manifold, the twistor space acquires an almost complex structure.

The thesis describes *all* 'germs' of twistor spaces of a Riemann surface and shows, with an example, that we can also consider compact generalised twistor spaces in the symplectic framework.

Conditions for maps into the twistor space to be holomorphic are given, following equivalent results from [22] in the case of $f$-twistors. A certain map is proved to be a biholomorphism between two twistor spaces if, and only if, the given connections are the affine transform of one another.

The riemannian geometry of twistor space with its natural hermitian metric is considered and taken forward, somehow corroborating the complex geometry. Later we find the best degree, in general, of holomorphic completeness of the twistor space. This leads also to an important aspect of a 'Penrose transform' in the symplectic case.

The thesis also gives some contribution to the theory of symplectic connections in showing a new proof of the deduction of which symplectomorphisms transform a flat translation invariant symplectic connection onto the trivial one over $(\mathbb{R}^{2n},\omega)$. This corresponds to a system of partial differential equations.

An exposition of modern differential geometry is given in the first chapter.

# Contents

# Introduction

In order to have a good understanding of the questions raised in this thesis, we think that a good amount of main ideas from the vast field of differential geometry must be fairly well explained. That is the objective of chapter 1. We also wish to introduce the reader, if not to our interpretation of the theory, then to the way we have thought we could better make use of it.

However, we consider that we may skip some of the less trivial theories, or rather the task of writing them down, such as a description of Lie groups, Lie derivatives and principal bundles, or an introduction to symplectic manifolds, complex manifolds and coherent sheaves, or even an exposition of the representation theory involved in our work — which, certainly, would be all very useful in reading the rest of the text. The reason is that such exposition could somehow affect the main purpose of the thesis: the presentation of new results on the geometry of twistor spaces and their proofs. Moreover, our notation is standard and the given bibliography is extensive in respect to the former. In chapter 1 we are thus not fully satisfied with the clarity of some deductions! Nevertheless, other proofs seem or are new, so not *all* of the chapter appears to us as being merely serving the rest of the thesis.

The latter is particularly true at the very end of chapter 1. Arising from earlier results in the theory, a system of (linear) differential equations is solved in order to conclude that certain translation invariant connections on $\mathbb{R}^{2n}$ are on the orbit of the trivial connection $\nabla^0$, *i.e.* they coincide with some affine transformation $\sigma \cdot \nabla^0$. Such specific result, which is already known, is kept here waiting to be interpreted in a wider context. For instance, the following could provide such a context: having found a new family $\mathcal{F}_1$ of symplectomorphisms we could now try to find those connections invariant for all elements of $\mathcal{F}_1$ and then compute the family $\mathcal{F}_2$ of symplectomorphisms connecting them to $\nabla^0$ or connecting them between each other, and so forth until some $\mathcal{F}_i = \emptyset$.

The present work began with the suggestion of Professor J. Rawnsley of looking carefully on to the twistor space of a symplectic manifold, a space which was considered the first time by I. Vaisman ([26]).

It is in chapter 2 that we present the theory of twistor spaces according to the seminal articles of N. O'Brian ([20]) and J. Rawnsley ([20, 22]). The twistor space of a symplectic manifold can be defined as a particular subset of the general twistor space, which is the bundle of all complex structures on the tangent spaces to any given manifold $M$ of even dimension. For the experienced reader, we recall that the dimensions of $Sp(2n,\mathbb{R})/U(n)$ and $SO(2n)/U(n)$ add up to the dimension of $GL(2n,\mathbb{R})/GL(n,\mathbb{C})$. So the symplectic twistor space, as I. Vaisman calls it, can also be viewed as a complement to another famous subspace, which is associated now to a given riemannian structure on the manifold and which refers to the original idea of twistors by R. Penrose.

More precisely, the riemannian and symplectic twistor spaces are the bundles of compatible complex structures, where compatible means orientation preserving linear isometric in the first case and linear symplectic in the second case. Moreover, the name 'twistor' is only really given to the bundle of complex structures when this is seen together with an almost-complex structure say $\mathcal{J}^\nabla$, which is a 'only in its kind' structure to many respects. It arises with the help of a linear connection $\nabla$ on $M$.

In the symplectic case the standard fibre of the bundle is non-compact. It has $n+1$ connected components which appear as the highest dimension open cells in a decomposition of $Sp(n)/U(n)$. Along the thesis we prove what was already known or suspected: there is no difference between the complex geometry of the various components of symplectic twistor spaces, and we end up considering only a certain first component. On the other hand, having another compact and complex-fibre bundle in which to embed our space, in section 2.3 we answer the question of extending the twistor almost-complex structure to such bundle.

In a more general context, we prove that there do not exist *certain* compact 'non-constant' $\omega$-twistor spaces (see the same section 2.3), which were the most expected to exist. Nevertheless, we still were able to produce an example of a compact 'non-constant' $\omega$-twistor space. The problem of 1-point complex compactification still remains. In particular, the twistor space of $(S^2, \omega)$ plus one point is diffeomorphic to $P^2(\mathbb{C})$ (*cf.* theory of divisors in [14] and our proposition 3.5). But is there any biholomorphism?

A short notice on notation: throughout this thesis $P^n(\mathbb{C})$ denotes only the underlying real symplectic manifold. Also, the very few new definitions appear between " ".

It happens that in the symplectic case, even for torsion free connections, there is a huge space parametrizing the almost-complex structures $\mathcal{J}^\nabla$. Thus there is a hope that twistors may be used in finding symplectic connections — selected as those for which $\mathcal{J}^\nabla$ is integrable. This thesis tries to show a few of the possible *connections*.

The problem with which the author first met symplectic twistor spaces consists just in describing the, undoubtedly, most trivial example: $\mathbb{R}^2$ with canonical symplectic structure and its trivial connection. While searching for a satisfactory and final answer we found more results and examples. Things evolved to the study of the induced action of symplectomorphisms $\sigma \in Symp(M, \omega)$ on the twistor space. Via affine transformation of $\nabla$ into $\sigma \cdot \nabla$ we could then ask if there was a biholomorphism of the twistor space under the respective complex structures. The answer appears in theorem 3.1.

We hope that, in the future, it will be possible to see which close relations are there to the Teichmüller space of $M$, since many of the induced actions, as the above, have a correspondent in twistor space. Regretfully there is also no time in this thesis to see the applications to the theory of hamiltonian Lie group actions on symplectic manifolds. There are already a few straightforward statements which the reader may deduce from the results we show here.

We make a study of the classical geometry of the twistor space of a symplectic manifold. It is possible to define a natural hermitian metric on it and we conclude that the twistor space is more likely to look negatively curved on the fibers and positively curved on its *horizontal* tangent planes. This is precisely what happens when the twistor space is kählerian, which we show to happen only to a flat $\nabla$.

If the almost-complex structure $\mathcal{J}^\nabla$ is integrable then we also find that, virtually, the twistor space of a $2n$-real symplectic manifold has a core compact $\mathbb{C}$-analytic submanifold of maximal dimension $n$: something like the graph of a holomorphic section of $M$. This is a consequence of holomorphic $n+1$-completeness. Now, by the principle of analytic continuation, we may conjecture that, if $j_1, j_2$ are two sections of the twistor bundle such that $j_i$ is $(j_i, \mathcal{J}^\nabla)$-holomorphic, for $i = 1, 2$, and such that $j_1 = j_2$ on some open subset of $M$, then the equality is globally true.

The geometrical results described in the previous paragraphs will appear in sections 3.2 and 3.3 and point again to new perspectives on the study of the possible different integrable complex structures of a fixed Kähler base manifold. However, if we are given two sections $j_1, j_2$ and if $j_1$ is holomorphic with respect to $(j_2, \mathcal{J}^\nabla)$, then we conclude that $j_1 = j_2$. This is a annoying corollary of section 2.4. The study in general of the twistor space of a Kähler manifold is another open problem.

We will sometimes call attention to some particular relation between the two subspaces sitting inside the general twistor space. However, if riemannian twistor spaces were invented in particular for the study of manifolds which do not possess a almost-complex structure, then the symplectic case has to serve a kind of manifold which has always many, but sometimes not a single preferred one.

In chapters 2 and 3 we thus present various results about the symplectic twistor space, which we consider the better the farther they are away from the other case — except in the point where we try to define a 'transform' in the $\overline{\partial}$-cohomology of the twistor space onto a sheaf of sections on the base space, in analogy to the

Penrose transform.

The latter led us to discover two new sheaves which are naturally defined over any Riemann surface — see section 3.4.

Finally, we advise that the reader eager to look for the new developments on the twistors of a symplectic manifold, having had a good experience with the up-to-date aspects of the theory, may go directly to chapter 3. There, we have put those results which gave us more trouble and which we consider our most independent work.

In the end there is also an appendix, both for verification of some particular results on representation theory of Lie groups and the completeness of this work.

# Chapter 1

## 1.1 Brief theory of connections

This first section is more for the author's sake than the readers' pleasure. The ones expert on the theory of connections are invited to proceed, assuming the usual standard notation as the one we shall establish here.

Let $E$ be a real or complex smooth vector bundle of rank $k$ over a paracompact smooth manifold $M$. A real or complex connection on $E$ is an object which enables us to describe *parallel displacements* in $E$. It can be given, for a start, by a covariant derivative: a local operator

$$\nabla : \Gamma E \longrightarrow \Gamma(T^*M \otimes E)$$

satisfying

$$\nabla f\xi = (\mathrm{d}f)\xi + f\nabla\xi, \tag{1.1}$$

for $\xi \in \Gamma(U;E)$, $f \in \mathrm{C}^\infty_U$, $U$ open in $M$. The tensor product above may be taken within $\mathbb{C}$ when appropriate conditions are fullfield. Now, if $s : U \to E^k$ is a frame field, *i.e.* $s(x) = (s_1(x),\ldots,s_k(x))$ is a basis of $E_x$, $\forall x \in U$, then $\nabla$ induces a local connection 1-form $\omega_{_U}$ with values in $\mathfrak{gl}$ ($\mathbb{R}^k$ or $\mathbb{C}^k$ according to the connection). This 1-form is determined by the identity

$$\nabla s = s\omega_{_U}. \tag{1.2}$$

A section $\xi$ of $E$ is said to be parallel if $\nabla\xi = 0$. Given a path $\gamma : I \subset \mathbb{R} \to M$, taking local coordinates for $E$ and $M$, it is a consequence of the existence of solutions to the first order differential equation

$$\nabla \xi_{|\gamma} = 0$$

that there are always parallel sections along $\gamma$. Problems arise which would require a whole frame $s_U$ of parallel sections on $U$. If a collection of such frames exists on an open cover $\mathcal{U} = \{U\}$ of $M$, we say the connection is flat. This implies that the transition functions, defined by

$$s_V = s_U g_{UV}$$

on $U \cap V$, take constant values in $GL$. Indeed

$$0 = \nabla s_V = (\nabla s_U) g_{UV} + s_U \mathrm{d} g_{UV} = s_U \mathrm{d} g_{UV}.$$

Conversely, such condition on a collection $\{s\}_{\mathcal{U}}$ is sufficient too for the existence of a flat connection on $E$. The proof follows trivially from the next assertion.

The 1-forms $\omega_U$ determine themselves the covariant derivative if and only if, when we change to the frame $s_V = s_U g_{UV}$, we find that

$$s_V \omega_V \;\; = \;\; \nabla (s_U g_{UV}) = s_U \omega_U g_{UV} + s_U \mathrm{d} g_{UV},$$

or equivalently

$$\omega_V = g_{UV}^{-1} \omega_U g_{UV} + g_{UV}^{-1} \mathrm{d} g_{UV}. \tag{1.3}$$

Considerations as that on the flat case lead to the concept of the curvature $R^\nabla$ of $\nabla$. Extending the latter by Leibnitz rule to

$$\nabla : \Gamma(T^*M \otimes E) \longrightarrow \Gamma(\wedge^2 TM \otimes E),$$

the former is defined by $R^\nabla = R = \nabla \circ \nabla$. On a section $s_U$ of the frame bundle of $E$ one must have

$$\begin{aligned} R s_U &= \nabla (s_U \omega_U) \\ &= \nabla s_U \wedge \omega_U + s_U \mathrm{d}\omega_U \\ &= s_U (\omega_U \wedge \omega_U + \mathrm{d}\omega_U). \end{aligned}$$

This defines

$$\rho_U = \mathrm{d}\omega_U + \omega_U \wedge \omega_U, \tag{1.4}$$

and a formula for $\mathrm{d}\rho_U$, the so called Bianchi identity, follows straightforwardly:

$$\mathrm{d}\rho_U = \rho_U \wedge \omega_U - \omega_U \wedge \rho_U .$$

Moreover, with no difficulty one proves both that $R$ is a differential 2-form with values in $\operatorname{End} E$, confirming the transformation of $\rho_U$ in

$$\rho_V = g_{UV}^{-1} \rho_U g_{UV}$$

just as a tensor, and that

$$R(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi$$

for any $X, Y \in \Gamma(TM) = \mathfrak{X}_M$. Following the latter remark on notation, from now on $\mathrm{A}^p(E)$ stands for $\Gamma(M; \wedge^p T^*M \otimes E)$ as it is usual.

We are glad to recall that (1.3) and (1.4) are part of E. Cartan's structure equations.

We shall also be using another characterisation of a connection. We fix a $k$-dimensional real or complex vector space $E_0$ and let $G = GL(E_0)$, and $\pi_0 : P \to M$ be the principal $G$-bundle of the frames of $E$,

$$P = \Big\{ p : E_0 \to E_x \text{ linear isomorphism : } \ x \in M \Big\}.$$

Recall that we have been assuming $\mathbb{R}$ or $\mathbb{C}$-linearity according to the connection, but, soon, this difference will lose its relevance as it will be resolved within the theory of connections itself. Now we want to see that $\nabla$ determines uniquely a global 1-form on the manifold $P$ with values in $\mathfrak{gl}(E_0)$.

First let $U$ be open in $M$ and $s_U : U \to P$ a section. Then $s_U$ induces a chart of $\pi_0^{-1}(U)$

$$\begin{aligned} \phi_U : \ U \times G &\longrightarrow \pi_0^{-1}(U) \\ (x,g) &\longmapsto s_U(x)g. \end{aligned}$$

If $(V, s_V)$ is another section with $s_V = s_U g_{UV}$ on $U \cap V$, then

$$\begin{aligned} \phi_U^{-1} \circ \phi_V(x, g) &= \phi_U^{-1}(s_V g) \\ &= \phi_U^{-1}(s_U g_{UV} g) \;=\; (x, g_{UV}(x) g). \end{aligned}$$

Omitting the projection to $G$, this is the same as $\phi_U^{-1}(p) = g_{UV} \circ \pi_0(p)\phi_V^{-1}(p)$. Denoting by $\varphi_U = \mathrm{pr}_2 \circ \phi_U^{-1}$ such projection and applying Leibnitz rule, we get

$$\mathrm{d}\varphi_U = \mathrm{d}(g_{UV} \circ \pi_0)\varphi_V + (g_{UV} \circ \pi_0)\mathrm{d}\varphi_V.$$

Now suppose we were given a connection on $E$, *i.e.* a collection of 1-forms $\{\omega_U\}$ satisfying the precise equations stated earlier (1.2,1.3). For $g \in G$, let $R_g$ be right multiplication by $g$ in $P$.

**Proposition 1.1.** *The $\mathfrak{gl}$-valued 1-form $\alpha$ defined on $\pi_0^{-1}(U)$ by*

$$\alpha = \mathrm{Ad}\,(\varphi_U^{-1})\pi_0^*\omega_U + \varphi_U^{-1}\mathrm{d}\varphi_U$$

*($\varphi_U^{-1}$ is the inverse inside $G$) does not depend on the choice of the maps $\varphi_U$ and hence gives rise to a 1-form on $P$. Moreover, $R_g^*\alpha = \mathrm{Ad}\,(g^{-1})\alpha$.*

*Proof.* Let us check the final result first. Using the relations $\varphi_U(s_U g) = g$, $\varphi_U \circ R_g = R_g \circ \varphi_U$ and $\pi_0 \circ R_g = \pi_0$, we deduce, on a point $p = s_U(x)g$ and a vector $Y$,

$$\begin{aligned} R_h^*\alpha_p(Y) &= \alpha_{ph}(\mathrm{d}R_{h\,p}Y) \\ &= (gh)^{-1}\pi_0^*\omega_{U\,ph}(\mathrm{d}R_{h\,p}Y)gh + (gh)^{-1}\mathrm{d}\varphi_{U\,ph}(\mathrm{d}R_{h\,p}Y) \\ &= h^{-1}g^{-1}(R_h^*\pi_0^*\omega_U)_p(Y)gh + h^{-1}g^{-1}\mathrm{d}(R_h \circ \varphi_U)_p(Y) \\ &= h^{-1}\left(g^{-1}(\pi_0^*\omega_U)_p(Y)g + g^{-1}\mathrm{d}\varphi_{U\,p}(Y)\right)h \\ &= \mathrm{Ad}\,(h^{-1})\alpha_p(Y). \end{aligned}$$

Now, to see that $\alpha$ is globally defined (compatible with the trivialisations of the principal bundle) we may already assume $p = s_V(x) = s_U(x)g(p)$, with $g(q) = g_{UV} \circ \pi_0(q)$. On one hand

$$\alpha = \pi_0^*\omega_V + \mathrm{d}\varphi_V$$

because $\varphi_{_V}(s_{_V}) = 1$, and on the other

$$\begin{aligned}\alpha &= g^{-1}\pi_0^*\omega_{_U} g + g^{-1}\mathrm{d}\varphi_{_U}\\ &= g^{-1}\left(g\pi_0^*\omega_{_V} g^{-1} - \mathrm{d}g\, g^{-1}\right) g + g^{-1}\left(\mathrm{d}g\,\varphi_{_V} + g\mathrm{d}\varphi_{_V}\right)\\ &= \pi_0^*\omega_{_V} + \mathrm{d}\varphi_{_V}.\end{aligned}$$

■

To better understand the *connection form* $\alpha$ we must take a quick look at the fundamental vector fields on the principal bundle $P$. Recall that an element $A \in \mathfrak{g}$, where $\mathfrak{g} = \mathfrak{gl}(E_0)$, induces a 1-parameter family of diffeomorphisms of $P$ by $R_{a_t}(p) = p\exp(tA)$ and that, in turn, this family determines a vector field on $P$ by

$$\tilde{A}_p = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} R_{a_t}(p),$$

where $a_t = \exp(tA)$.

**Lemma 1.1.** *The map from $\mathfrak{g}$ to $\mathfrak{X}_P$ which sends $A$ to the fundamental vector field $\tilde{A}$ is a Lie algebra monomorphism.*

*Proof.* Of course $\tilde{A}(f)$ is a smooth function for any $f \in \mathrm{C}_P^\infty$, since the action of $G$ in $P$ is itself smooth by construction, and this is sufficient to prove $\tilde{A}$ is smooth. Now

$$\begin{aligned}\mathrm{d}R_{_g}(\tilde{A}_p) &= \frac{\mathrm{d}}{\mathrm{d}t} R_{_g} \circ R_{a_t}(p)\\ &= \frac{\mathrm{d}}{\mathrm{d}t} pg g^{-1} a_t g \;=\; {}^{\sim}(\mathrm{Ad}\,(g^{-1})A)_{pg}\end{aligned}$$

because $\exp\left(t\mathrm{Ad}\,(g)A\right) = g a_t g^{-1}$. Hence, up to the action on smooth functions,

$$\begin{aligned}[\tilde{A}, \tilde{B}] &= \lim_{t\to 0} \frac{1}{t}\left(\tilde{B} - \mathrm{d}R_{a_t}(\tilde{B})\right)\\ &= {}^{\sim}\lim_{t\to 0}\frac{1}{t}\left(B - \mathrm{Ad}\,(a_t^{-1})B\right) \;=\; \widetilde{[A, B]},\end{aligned}$$

since the continuity of $^{\sim}$ is immediate. ■

For sign matters, notice the Lie derivative above is the one which induces $[X, Y] = XY - YX$, for all $X, Y \in \mathfrak{X}$.

Clearly one can see that the fundamental vector fields are tangent to the fibres of $P$. Indeed, they generate $\mathcal{V} = \ker \mathrm{d}\pi_0$ — the vertical distribution. Also, going back to the connections,

$$\begin{aligned} \alpha(\tilde{A}) &= \varphi_{U}^{-1}(s_{U}) \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \varphi_{U}(s_{U} \exp tA) \\ &= 1 \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \exp tA \ = \ A, \end{aligned}$$

on any point $p = s_{U}(x)$. If $v : U \to E_0$ is any function, from which we get a section of $E = P \times_G E_0$, then

$$\nabla s_{U} v = s_{U}(s_{U}^{*}\alpha)\, v + s_{U} \mathrm{d}v \tag{1.5}$$

and we have almost proved the next proposition.

**Proposition 1.2.** *A $\mathfrak{g}$-valued 1-form on $P$ satisfying $\alpha(\tilde{A}) = A$ on fundamental vector fields and $R_{g}^{*}\alpha = \mathrm{Ad}\,(g^{-1})\alpha$ determines uniquely a connection on $E$.*

*Proof.* We are left to show that $\nabla$ defined in (1.5) satisfies (1.3) with $\omega_{U} = s_{U}^{*}\alpha$. So suppose $s_{U}, s_{V}$ are two frame fields and $s_{V} = s_{U} g_{UV}$. Consider the map $k : P \times G \to P,\ k(p, g) = pg$. Then $s_{V} = k \circ (s_{U}, g_{UV})$, so

$$\mathrm{d}s_{V} = \mathrm{d}k(\mathrm{d}s_{U}, \mathrm{d}g_{UV}) = \mathrm{d}R_{g_{UV}} \circ \mathrm{d}s_{U} + \mathrm{d}L_{s_{U}} \circ \mathrm{d}g_{UV},$$

where $L_{p} : G \to P$ has the obvious meaning. Now, for $Z_g \in T_gG$,

$$\begin{aligned} \alpha_{pg}(\mathrm{d}L_{p}(Z_g)) &= \alpha_{pg}\left(\mathrm{d}L_{p}\left(g \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \exp t g^{-1} Z_g\right)\right) \\ &= \alpha_{pg}\left(\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} pg \exp t g^{-1} Z_g\right) \\ &= g^{-1} Z_g. \end{aligned}$$

Hence, this time using the second hypothesis on $\alpha$,

$$\begin{aligned} s_{V}^{*}\alpha &= R_{g_{UV}}^{*} \alpha(\mathrm{d}s_{U}(\cdot)) + \alpha\left(\mathrm{d}L_{s_{U}}(\mathrm{d}g_{UV}(\cdot))\right) \\ &= \mathrm{Ad}\,(g_{UV}^{-1}) s_{U}^{*}\alpha + g_{UV}^{-1}\mathrm{d}g_{UV}, \end{aligned}$$

as we wished. ∎

Naturally one calls parallel those (local) sections for which $s_* TM \subset \ker \alpha$ — just see what happens in formula (1.5) when $v$ is constant. The smooth subvector bundle $\mathcal{H} = \ker \alpha$ is called the horizontal distribution, and it is in fact a complement for $\mathcal{V}$ in $TP$: again, by ordinary differential equations, we can never avoid them once, there exists a parallel section along any curve in $M$, with given initial point in $P$ and initial tangent direction in $\mathcal{H}$ — a *horizontal lift* of the curve. Since all sections are embeddings, one proves the assertion by counting dimensions. From proposition 1.2 follows that $\mathcal{H}_{pg} = \mathrm{d}R_{_g}\mathcal{H}_p$. Reciprocally, we can understand now that a smooth family of subspaces $\mathcal{H}$, complementary to the vertical distribution and satisfying the previous right-invariance, determines a unique connection.

If the structure group of $E$ is reducible, and hence $E$ is associated to a principal $G$-subbundle $Q \subseteq P$, $G \subseteq GL(k)$, then the connection is said to be reducible to $Q$ if $\mathcal{H}_{|Q} \subset TQ$. The whole theory remains just the same in case such restriction occurs, so from now on we assume $P$ and $G$ in that kind of generality. The first example is the reduction from $GL(k,\mathbb{R})$ to $GL(k/2,\mathbb{C})$, the two cases we had been dealing together. To see that the inclusion above is equivalent to $\nabla$ being $\mathbb{C}$-linear one must reason with differentiable curves. Before we go farther, notice that had we started with a $H$-connection on $Q$, proposition 1.2 would have led us to a unique $G$-connection *extension* on $P$.

The $\mathfrak{g}$-valued 2-form

$$\rho = \mathrm{d}\alpha + \alpha \wedge \alpha \tag{1.6}$$

is called the curvature of $\alpha$. This is because, for any section $s : U \to P$, $s^*\alpha = \omega_{_U}$ and hence $s^*\rho = \rho_{_U}$, a $\mathfrak{g}$-valued form too (*cf.* (1.4)). Let us see other remarkable properties.

**Proposition 1.3.** *(i) For* $A \in \mathfrak{g}$ *and* $X \in \Gamma\mathcal{H}$*, we have* $[\tilde{A}, X] \in \Gamma\mathcal{H}$.
*(ii)* $\rho(\mathcal{V}, \cdot) = 0$.
*(iii)* $\rho$ *is identically* 0 *iff* $\Gamma\mathcal{H}$ *is an integrable distribution.*

*Proof.* *(i)* Let $a_t = \exp tA$. Then, omitting the action on functions,

$$[\tilde{A}, X] = \lim_{t\to 0} \frac{1}{t}\Big(X - \mathrm{d}R_{a_t}(X)\Big) \in \mathcal{H}$$

since the distribution is closed in $TP$.

*(ii)* Let $\tilde{A} \in \Gamma\mathcal{V},\ Y \in \mathfrak{X}_P$. By tensoriality we may suppose $\tilde{A}$ and $\tilde{B}$, the vertical part of $Y$, are fundamental vector fields. Then

$$\begin{aligned}
\rho(\tilde{A}, Y) &= \tilde{A}\cdot\alpha(Y) - Y\cdot\alpha(\tilde{A}) - \alpha[\tilde{A}, Y] + \alpha\wedge\alpha(\tilde{A}, Y) \\
&= \tilde{A}\cdot B - Y\cdot A - \alpha[\tilde{A}, \tilde{B}] + \alpha\wedge\alpha(\tilde{A}, \tilde{B}) \\
&= -\alpha\widetilde{[A, B]} + [A, B] = 0.
\end{aligned}$$

*(iii)* For $X, Y$ horizontal, $\rho(X, Y) = -\alpha[X, Y]$. Then we use Frobenius integrability theorem. ■

From the proposition we conclude that a connection is flat if and only if any, and hence all, of the 2-forms $R^\nabla$, all the $\rho_U$ or $\rho$ are zero.

**Theorem 1.1.** *A connection is flat iff its curvature is* 0*.*

To finish the present section we recall that connections do exist on every principal $G$-bundle over a paracompact manifold. Moreover, the space of all connections on $E$ is an affine space modelled in $\mathrm{A}^1(\mathrm{End}\, E)$, as it is easy to show.

A necessary note on bibliography: inspiration for all the above came from various sources like [11, 13, 16, 17].

## 1.2 Other connections

Assume here all the setting from the last section. By proposition 1.2 we see that it is possible to construct connections on $\mathrm{End}\, E$, $\wedge^p E$, $E^*$, etc, from a given connection $\nabla$ on $E \to M$, just replacing $E_0$ by the appropriate linear representation space of $G$ and composing $\alpha$ with the induced representation of $\mathfrak{g}$. Taking another vector bundle $F$ over $M$ with another connection $\nabla^1$, we can also combine (representations) $\nabla, \nabla^1$ to get connections in $E \oplus F$ or $E \otimes F$. Formulas for their curvatures follow with a little work, for example

$$R^{\nabla\otimes\nabla^1} = R^{\nabla} \otimes \mathrm{Id} + \mathrm{Id} \otimes R^{\nabla^1},$$

and everything is in coherence with the trivial connection on $M \times \mathbb{R}$.

If $Y$ is any homogeneous effective $G$-space, for each $y_0 \in Y$ there is a sequence

$$P \longrightarrow Z \longrightarrow M$$

of fibre bundles (in the sense of [14]) with composition $\pi_0$, where $Z = P \times_G Y$. This induces a horizontal distribution in $TZ$, which is integrable if the connection is flat — a fact which could be not true, were there not $G$-equivariance of the initial distribution $\mathcal{H}$ over $P$.

One can use the previous ideas and go farther to prove, for example, that given a metric $g$ in $E$ the connection will be reducible to the principal $O(k)$-bundle of orthonormal frames if, and only if, $\nabla g = 0$, using now the induced connection on $\otimes^2 E^*$. The same is true for a hermitian or a symplectic structure.

The latter will play a fundamental role in this thesis, so we recall that if $(E, \omega)$ is a symplectic vector bundle and $\nabla\omega = 0$, then $\nabla$ is called a symplectic connection — with the exception only of the tangent bundle to a symplectic manifold, in the case of which, as we shall see, an extra assumption is required.

Let $\sigma : N \to M$ be a smooth map. One can define a new connection $\sigma^*\nabla$ on

$\sigma^* E \to N$ by

$$\sigma^* \nabla_X \sigma^* \xi = \sigma^* (\nabla_{\mathrm{d}\sigma(X)} \xi),$$

where $\sigma^* \xi = \xi \circ \sigma$, for any $\xi \in \Gamma E$. Clearly the sections $\sigma^* \xi$ generate $\Gamma \sigma^* E$ as a $\mathrm{C}^\infty_N$-module. The connection will be referred to as the pull-back connection. It is easy to prove

$$R^{\sigma^* \nabla}(X, Y) = \sigma^* R^\nabla (\mathrm{d}\sigma\, X, \mathrm{d}\sigma\, Y), \tag{1.7}$$

for any $X, Y \in TN$. Notice the implicit identification of $\sigma^* \mathrm{End}\, E$ with $\mathrm{End}\, \sigma^* E$. Further we remark that, starting in this section, this formula will be used in many situations until the end of our thesis.

We will also need the following differential operator $\mathrm{d}^\nabla : \mathrm{A}^q(E) \to \mathrm{A}^{q+1}(E)$ defined in degree 0 by $\mathrm{d}^\nabla \xi = \nabla \xi$, and in general by

$$\begin{aligned} \mathrm{d}^\nabla \xi(X_0, \ldots, X_q) &= \sum_i (-1)^i \nabla_{X_i} \xi(X_0, \ldots, X_{i-1}, X_{i+1}, \ldots, X_q) \\ &\quad + \sum_{i<j} (-1)^{i+j} \xi([X_i, X_j], X_0, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_q) \end{aligned}$$

$\forall X_i \in TM$. This extension of the de Rham complex through the connection satisfies the same sign rules over $\mathrm{A}^*$-left multiplication, and had already appeared in section 1.1 defining $\mathrm{d}^\nabla \circ \mathrm{d}^\nabla = R^\nabla$. Seeing the latter as an element of $A^2(\mathrm{End}\, E)$ and differentiating as above we get

$$\mathrm{d}^\nabla R^\nabla = 0,$$

which is Bianchi identity again.

Now, consider a multilinear form

$$f^i : \times_{j=1}^i \mathrm{End}\, E_0 \longrightarrow \mathbb{R} \ \text{ or } \ \mathbb{C},$$

with respective real or complex endomorphisms according to the connection. Let $f^i$ be $G$-invariant, *i.e.* $f^i(gA_1 g^{-1}, \ldots, gA_i g^{-1}) = f^i(A_1, \ldots, A_i)$, $\forall g \in G$. From now on, if we denote a form $Q \in \mathrm{A}^p(\mathrm{End}\, E)$ by

$$Q = \beta B$$

then we mean $\beta \in \mathrm{A}^p$ and $B \in \mathrm{A}^0(\mathrm{End}\, E)$. Locally, every $Q$ is decomposable in many ways as a sum of forms of that type.

With simple $Q_j = \beta_j B_j$, $j = 1, \ldots, i$, we define a new multilinear map, also denoted $f^i$, by

$$f^i(Q_1, \ldots, Q_i) = \beta_1 \wedge \ldots \wedge \beta_i \, f^i(\hat{B}_1, \ldots, \hat{B}_i),$$

where $\hat{B}_j$ is $s^{-1}B_j s : U \to \mathrm{End}\, E_0$ and $s : U \to P$ is any local section. Extending linearly,

$$f^i : \otimes^i \mathrm{A}^*(\mathrm{End}\, E) \longrightarrow \mathrm{A}^*$$

is surely well defined — a 0-degree local operator if one pleases.

**Lemma 1.2.** *Let* $p_0 = 0$. *Then*

$$\mathrm{d}(f^i(Q_1, \ldots, Q_i)) \;=\; \sum_j (-1)^{p_1 + \ldots + p_{j-1}} f^i(Q_1, \ldots, \mathrm{d}^\nabla Q_j, \ldots Q_i).$$

*Proof.* In 0-degree we have

$$\begin{aligned}
\mathrm{d}f^i(B_1, \ldots, B_i) &= \sum_j f^i(\hat{B}_1, \ldots, \mathrm{d}\hat{B}_j, \ldots, \hat{B}_i) \\
&= \sum_j \Big\{ f^i(\hat{B}_1, \ldots, \mathrm{d}\hat{B}_j, \ldots, \hat{B}_i) + f^i(\hat{B}_1, \ldots, [s^*\alpha, \hat{B}_j], \ldots, \hat{B}_i) \Big\} \\
&= \sum_j f^i(B_1, \ldots, \mathrm{d}^\nabla B_j, \ldots, B_i).
\end{aligned}$$

The second term on the second line, the sum equal to 0, comes from an equivalent definition of $G$-invariance for multilinear maps at the Lie algebra level (differentiating the adjoint action $\mathrm{Ad}\,(\exp tA)$ on $f^i$). For the $Q_j = \beta_j B_j$,

$$\begin{aligned}
\mathrm{d}(f^i(Q_1, \ldots, Q_i)) = \sum_j \Big\{ &(-1)^{p_1 + \ldots + p_{j-1}} \beta_1 \wedge \ldots \wedge (\mathrm{d}\beta_j) \wedge \ldots \\
&\ldots \wedge \beta_i \Big\} f^i(B_1, \ldots, B_i) + (-1)^{p_1 + \ldots + p_i} \beta_1 \wedge \ldots \wedge \beta_i \wedge \mathrm{d}(f^i(B_1, \ldots, B_i))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_j \Big\{ (-1)^{p_1+\ldots+p_{j-1}} f^i(Q_1, \ldots, \mathrm{d}\beta_j\, B_j, \ldots, Q_i) \\
&\qquad + (-1)^{p_1+\ldots+p_i} \beta_1 \wedge \ldots \wedge \beta_i \wedge f^i(B_1, \ldots, \mathrm{d}^\nabla B_j, \ldots, B_i) \Big\} \\
&= \sum_j \Big\{ (-1)^{p_1+\ldots+p_{j-1}} f^i(Q_1, \ldots, \mathrm{d}\beta_j\, B_j, \ldots, Q_i) \\
&\qquad + (-1)^{p_1+\ldots+p_{j-1}} f^i(Q_1, \ldots, (-1)^{p_j} \beta_j \wedge \mathrm{d}^\nabla B_j, \ldots, Q_i) \Big\},
\end{aligned}
$$

which equals the stated formula. The general case follows by linearity. ■

Let $f^i(Q) = f^i(Q, \ldots, Q)$. As it is well known, certain *symmetric* multilinear forms give origin to the Chern classes of the vector bundle in the complex setting (*cf.* [16]). For general forms we still have the following.

**Proposition 1.4.** *$f^i(R^\nabla)$ is closed and $c^{f^i} = [f^i(R^\nabla)] \in H^{2i}(M)$ does not depend on the choice of the connection.*

*Proof.* By the lemma above and the Bianchi identity we get the first part. For the second, we use the same proof shown in [16] for symmetric $f^i$. Let $\nabla^t = \nabla^0 + tA$, a ray of connections in the affine space. In the following we use the notation $\nabla$ for $\mathrm{d}^\nabla$ on $E$-valued forms and reserve $\mathrm{d}^\nabla$ for the $\mathrm{End}\, E$ ones. For $e \in E$,

$$
\begin{aligned}
R^{\nabla^t} e &= (\nabla^0 + tA)(\nabla^0 e + tAe) \\
&= R^{\nabla^0} e + t(\mathrm{d}^{\nabla^0} A)e + t^2 (A \wedge A)e
\end{aligned}
$$

because, if we assume shortly $A = \alpha_j A_j$, then

$$
\begin{aligned}
(\mathrm{d}^{\nabla^0} A)e &= \mathrm{d}\alpha_j A_j e - \alpha_j \wedge (\mathrm{d}^{\nabla^0} A_j)e \\
&= \mathrm{d}\alpha_j A_j e - \alpha_j \wedge \nabla^0(A_j e) + \alpha_j \wedge A_j(\nabla^0 e) \\
&= \nabla^0(\alpha_j A_j e) + \alpha_j A_j \wedge \nabla^0 e \\
&= \nabla^0(Ae) + A \wedge \nabla^0 e,
\end{aligned}
$$

the equation one would expect. An extra line of work, using the identity just found, but now for $\nabla^t$, proves

$$
\mathrm{d}^{\nabla^t} A = \mathrm{d}^{\nabla^0} A + 2tA \wedge A.
$$

Hence $\partial R^{\nabla^t}/\partial t = \mathrm{d}^{\nabla^t} A$. Finally, lighting the notation,

$$\begin{aligned} f^i(R^{\nabla^1}) - f^i(R^{\nabla^0}) &= \int_0^1 \frac{\partial}{\partial t} f^i(R^t, \dots, R^t)\mathrm{d}t \\ &= \int_0^1 \sum_{j=1}^{i} f^i\left(R^t, \dots, \frac{\partial R^t}{\partial t}, \dots, R^t\right)\mathrm{d}t \\ &= \mathrm{d}\left(\sum \int f^i(R^t, \dots, A, \dots, R^t)\mathrm{d}t\right) \end{aligned}$$

by lemma 1.2. ■

For example, the classes induced by the form (non-symmetric for $i > 2$)

$$\begin{aligned} \mathrm{Tr} \ : \ \mathfrak{g} \times \cdots \times \mathfrak{g} &\longrightarrow \mathbb{R} \\ (B_1, \dots, B_i) &\longmapsto \mathrm{Tr}\,(B_1 \circ \cdots \circ B_i) \end{aligned}$$

on the tangent bundle, associate a *topological charge*, a number, to every compact $2i$-manifold (*cf.* [11]). We remark that different representations $\mathfrak{g}$ of End may be assumed in the literature.

Just by looking at the definitions and formula (1.7) we get the following result.

**Proposition 1.5.** *If $\sigma$ is a smooth map between two manifolds, then*

$$c^{f^i}(\sigma^* E) = \sigma^*\, c^{f^i}(E).$$

If $f_1^i, f_2^j$ are multilinear $G_1, G_2$-invariant forms in $\times^i \mathrm{End}\, E_{0_1}$ and $\times^j \mathrm{End}\, E_{0_2}$, respectively, then the form $f_1^i \times f_2^j$ extends to a $G_1 \times G_2$-invariant multilinear form on

$$\times^{i+j} \mathrm{End}\,(E_{0_1} \oplus E_{0_2}).$$

The extension is $f_1^i \times f_2^j(A_1, \dots, A_{i+j}) = f_1^i(A_1, \dots, A_i) f_2^j(A_{i+1}, \dots, A_{i+j})$, where the $A_l$ on the right hand side are factored through the natural short exact sequences

$$0 \longrightarrow E_{0_r} \longrightarrow E_{0_1} \oplus E_{0_2} \longrightarrow E_{0_{r+1}} \longrightarrow 0$$

($r \bmod 2$). Since $R^{\nabla^1 \oplus \nabla^2} = R^{\nabla^1} \oplus R^{\nabla^2}$ and thus preserves the direct sum, we see by the definitions that

$$f_1^i \times f_2^j(R^{\nabla^1 \oplus \nabla^2}) = f_1^i(R^{\nabla^1}) \wedge f_2^j(R^{\nabla^2}),$$

which leads us to the next proposition.

**Proposition 1.6.** $c^{f_1^i\times f_2^j}(E_1\oplus E_2)=c^{f_1^i}(E_1)c^{f_2^j}(E_2)$.

We recall that there exist families $\{f^i\}_{i\geq 0}$ of generically defined forms for which the *total* class $c=\sum_i c^{f^i}$ verifies $c(E_1\oplus E_2)=c(E_1)c(E_2)$. For example, the total Chern class or the Chern character. We could go on now to the case of tensor products, but this would take us far away from our desired course of studies.

Each module $\mathrm{A}^p(\otimes^l E)$ has its own differential operator $\mathrm{d}^\nabla$, but if we combine the wedge product on forms with tensor product of $E$ and $F$, both vector bundles endowed with a connection, and denote again the resulting associative product by $\wedge$, then we find the following relation.

**Lemma 1.3.** *Let* $P\in\mathrm{A}^p(E),\ \ Q\in\mathrm{A}^q(F)$. *Then*

$$\mathrm{d}^\nabla(P\wedge Q)=\mathrm{d}^\nabla P\wedge Q+(-1)^pP\wedge\mathrm{d}^\nabla Q.$$

*Proof.* We may already assume $P=\alpha\, e,\ Q=\beta\, f$. Then

$$\begin{aligned}
&\mathrm{d}^\nabla(\alpha\wedge\beta(e\otimes f))\\
&= \mathrm{d}(\alpha\wedge\beta)e\otimes f+(-1)^{p+q}\alpha\wedge\beta\,\nabla(e\otimes f)\\
&= (\mathrm{d}\alpha\wedge\beta+(-1)^p\alpha\wedge\mathrm{d}\beta)e\otimes f\\
&\qquad\qquad +(-1)^{p+q}\alpha\wedge\beta(\nabla e\otimes f+e\otimes\nabla f)\\
&= (\mathrm{d}\alpha\, e)\wedge\beta\, f+(-1)^p\alpha\, e\wedge(\mathrm{d}\beta\, f)\\
&\qquad\quad +(-1)^p\alpha\wedge\nabla e\wedge\beta\, f+(-1)^p\alpha\, e\wedge(-1)^q\beta\wedge\nabla f\\
&= \mathrm{d}^\nabla(\alpha\, e)\wedge(\beta\, f)+(-1)^p(\alpha\, e)\wedge\mathrm{d}^\nabla(\beta\, f),
\end{aligned}$$

the formula we searched. ■

We note that restricting to a second wedge product induced by

$$\wedge^{l_1}E\ \wedge\ \wedge^{l_2}E\longrightarrow\wedge^{l_1+l_2}E$$

the formula above is still valid, as there is no sign rule in covariant differentiation of wedge products. Notice however the identity

$$P_1 \wedge P_2 = (-1)^{p_1p_2+l_1l_2} P_2 \wedge P_1, \tag{1.8}$$

for $P_i \in \mathrm{A}^{p_i}(\wedge^{l_i}E)$.

Suppose now that we are in the presence of a linear *or* affine connection[1], *i.e.* defined on the tangent bundle of $M$. The 2-tensor $\mathrm{d}^\nabla \mathrm{Id} = T^\nabla = T$ is very important and known as the torsion of the connection. We see $\mathrm{Id} \in \mathrm{A}^1(TM)$. Trivially,

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

Let us analyze this at the level of the principal bundle $FM$ of tangent basis to $M$. Let $\theta$ be the $E_0$-valued 1-form on $FM$ (the rank $k$ is now the dimension of the manifold)

$$\theta_p(Z) = p^{-1}\mathrm{d}\pi_0(Z), \tag{1.9}$$

where $\pi_0 : FM \to M$ is the projection map.

**Proposition 1.7.** *The torsion tensor of* $\nabla$ *corresponds to the form*

$$\tau = \mathrm{d}\theta + \alpha \wedge \theta$$

*on* $FM$.

*Proof.* Let $s : U \to FM$ be a section and $X, Y \in \mathfrak{X}_U$. If $X = sv$, with $v : U \to E_0$ smooth, then we have $s^*\theta(X) = v$. Hence, by formula (1.5),

$$\begin{aligned} T(X,Y) &= s\,s^*\alpha(X)s^*\theta(Y) + s\,\mathrm{d}(s^*\theta(Y))(X) \\ &\quad -s\,s^*\alpha(Y)s^*\theta(X) - s\,\mathrm{d}(s^*\theta(X))(Y) - s\,s^*\theta[X,Y] \\ &= s\,\left(s^*(\alpha\wedge\theta)(X,Y) + \mathrm{d}s^*\theta(X,Y)\right) \\ &= s\,s^*(\alpha\wedge\theta + \mathrm{d}\theta)(X,Y). \end{aligned}$$

∎

[1]There seems to have been proved that every $GL(n,\mathbb{R}) \rtimes \mathbb{R}^n$-connection is reducible to a $GL(n,\mathbb{R})$-connection.

In the context of a riemannian manifold $(M, g)$ the fundamental theorem of riemannian geometry states that there exists a unique torsion free, metric connection, the Levi-Civita connection, which we shall denote by $\nabla^g$ and assume to be very well known to all. For an almost-hermitian manifold $(M, h, J)$ there exist, theoretically, many hermitian linear connections: making both $h$ and $J$ parallel.

If $g$ is any tensor of degree $p$, then we can see it as

$$g \in \mathrm{A}^0(\otimes^p T^*M).$$

Induced by a canonical isomorphism, for each such $g$ we have another

$$\tilde{g} \in \mathrm{A}^0((\otimes^p TM)^*).$$

However,

$$\begin{aligned} \mathrm{d}^\nabla \tilde{g}\,(X_0)\,(X_1 \otimes \ldots \otimes X_p) = \big(\nabla_{X_0} \tilde{g}\big)(X_1 \otimes \ldots \otimes X_p) \\ = X_0(\tilde{g}(X_1 \otimes \ldots \otimes X_p)) - \tilde{g}\left(\nabla_{X_0}(X_1 \otimes \ldots \otimes X_p)\right) \\ = \nabla_{X_0} g\,(X_1, \ldots, X_p), \end{aligned}$$

so the isomorphism is parallel and we will not distinguish $g$ from $\tilde{g}$. Next we check that our definitions, in a special case, coincide with simple contraction.

**Lemma 1.4.** *For $Q \in \mathrm{A}^q(\otimes^p TM)$, we have*

$$\mathrm{Tr}\,(g\,Q) = g \circ Q.$$

*Proof.* The notation is expressing the role of $g\,Q$ as an element of $\mathrm{A}^q(\mathrm{End}\,(\otimes^p TM))$. Let $(x_1, \ldots, x_n)$ be a chart of $M$ and suppose

$$Q = q^{j_1 \ldots j_p}_{i_1 \ldots i_q} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_q} \frac{\partial}{\partial x^{j_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{j_p}}.$$

Then

$$\begin{aligned} \mathrm{Tr}\,(g\,Q) &= q^{j_1 \ldots j_p}_{i_1 \ldots i_q} \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_q}\, g\left(\frac{\partial}{\partial x^{l_1}}, \ldots, \frac{\partial}{\partial x^{l_p}}\right) \cdot \\ &\qquad \mathrm{d}x_{l_1} \otimes \ldots \otimes \mathrm{d}x_{l_p} \left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_p}}\right) \\ &= q^{j_1 \ldots j_p}_{i_1 \ldots i_q}\, g_{j_1 \ldots j_p}\, \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_q} \;=\; g(Q). \end{aligned}$$

■

After this simple lemma we are ready to present the result which was the motive to go through the largest part of the work in the present section and which has been in use in many ways since a long, long time ago.

**Proposition 1.8.** *For every* $\omega \in \mathrm{A}^p$,

$$\omega = \mathrm{Tr}\,(\omega\ \beta_p)$$

*where* $\beta_p = \frac{1}{p!}\wedge^p \mathrm{Id} \in \mathrm{A}^p(\wedge^p TM)$. *Furthermore*

$$\mathrm{d}\omega = \mathrm{Tr}\,(\nabla\omega\wedge\beta_p) + \mathrm{Tr}\,(\omega\ T^\nabla\wedge\beta_{p-1}).$$

*Proof.* It is simple to see $\beta_p(X_1,\ldots,X_p) = X_1\wedge\ldots\wedge X_p$, so the first part follows by the lemma above. To find $\mathrm{d}\omega$ we apply lemmas 1.2 and 1.3. and compute

$$\begin{aligned} p!\,\mathrm{d}^\nabla\beta_p &= T\wedge\mathrm{Id}^{p-1} - \mathrm{Id}\wedge T\wedge\mathrm{Id}^{p-2} + \mathrm{Id}^2\wedge T\wedge\mathrm{Id}^{p-3} - \ldots \\ &= p\,T\wedge\mathrm{Id}^{p-1}, \end{aligned}$$

using (1.8). ∎

Applying d repeatedly to the formula just found we get *generalised* Bianchi identities. In particular, when $T^\nabla = 0$, we find

$$\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\!\bigoplus} R^\nabla(X,Y)Z = 0,$$

inferred from the identity $0 = \mathrm{Tr}\,(R^\nabla\,\omega\wedge\mathrm{Id}),\ \forall\omega\in\mathrm{A}^1$. It is the former which is known as *the* Bianchi identity (*cf.* [13]).

A final remark: it seems one could have risked more to try to find results, like the ones proved in this section, for any vector bundle $E$ and not just for $\mathrm{End}\,E$. And also for vector bundle-valued multilinear forms, because we do not want to believe lemma 1.2 is merely a geometrical coincidence.

## 1.3 On holomorphic vector bundles

For the moment we assume a lot of the theory of complex manifolds. The reader will be asked, later, to recall in detail some of its basic foundations.

Suppose $E \to M$ is holomorphic and let $\mathcal{E}$ be the sheaf $\mathcal{O}(E)$. It is well known that there exists a unique local operator

$$\overline{\partial}_{\mathcal{E}} : \mathrm{A}^{p,q}(E) \longrightarrow \mathrm{A}^{p,q+1}(E)$$

determined by

$$\begin{cases} \overline{\partial}_{\mathcal{E}}(fe) = (\overline{\partial} f)e + f\overline{\partial}_{\mathcal{E}} e, \quad \text{for all } f \in \mathrm{C}^\infty,\ e \in \Gamma(U; \wedge^{p,q} \otimes E), \\ \overline{\partial}_{\mathcal{E}} e = 0 \text{ on an open subset } U \subset M \text{ iff } e \text{ is holomorphic on } U \end{cases}$$

(*cf.* [11]). Since $\overline{\partial}_{\mathcal{E}}^2 = 0$ and

$$\overline{\partial}_{\mathcal{E}}(\omega \wedge \theta) = \overline{\partial}\omega \wedge \theta + (-1)^p \omega \wedge \overline{\partial}_{\mathcal{E}}\theta,$$

$\forall \omega \in \mathrm{A}^{k,l},\ \theta \in \mathrm{A}^{p,q}(E)$, there is a proper ring cohomology theory

$$H^{p,*}(M, \mathcal{E}) = \ker \overline{\partial}_{\mathcal{E}} / \mathrm{Im}\, \overline{\partial}_{\mathcal{E}},$$

which by *locality* can unambiguously be described through the methods of the theory of sheaves. (Though already intensively mentioned, we are still on time to recall the concept of 'local operator': if two sections on $U$ agree in some open $V \subset U$ then their $\overline{\partial}_{\mathcal{E}}$ also agree.)

If $\phi : N \to M$ is a smooth map between complex manifolds, then we may define on $\phi^* E$ another local operator

$$\phi^* \overline{\partial}_{\mathcal{E}}$$

exactly in the same way as we defined the pull-back connection in the previous section. All we know is that, if $\phi$ is holomorphic, then by uniqueness $\overline{\partial}_{\phi^*\mathcal{E}} = \phi^* \overline{\partial}_{\mathcal{E}}$.

Let $',''$ denote, as it is usual, the decomposition of a 1-form according to type. A smooth hermitian structure $h$ on a holomorphic vector bundle determines a unique hermitian connection $D$, *i.e.* reducible to the principal bundle of unitary frames, such that $D'' = {}'' \circ D = \overline{\partial}_{\mathcal{E}}$: letting $s_{_U} = (s_1, \ldots, s_k)$ denote a local holomorphic frame field and $H$ the matrix $[h(s_i, s_j)]$ we must have

$$\mathrm{d}(h(s_i, s_j)) = h(Ds_i, s_j) + h(s_i, Ds_j),$$

or simply

$$\mathrm{d}H = \omega_{_U}^t H + H\overline{\omega}_{_U}.$$

Since $\omega_{_U}$ must be of type $(1,0)$,

$$\partial H = \omega_{_U}^t H \tag{1.10}$$

determines the connection 1-form and one checks equation (1.3) easily to get the required $D$. We remark that differentiating again in $\mathrm{d}H$ gives $\rho_{_U}^t H + H\overline{\rho}_{_U} = 0$. Clearly $R^{0,2} = D^{0,2} \circ D = 0$ and hence $\rho_{_U}$ is type $(1,1)$. Reciprocally and more generally, we found a proof of the following well known theorem due to Koszul and Malgrange.

**Theorem 1.2.** *Let $M$ be a complex manifold and let $E \to M$ be a complex vector bundle with a $\mathbb{C}$-linear connection $\nabla$ such that $R^{0,2} = 0$. Then $E$ admits a unique structure of a holomorphic vector bundle, such that $\nabla'' = \overline{\partial}_{\mathcal{E}}$.*

*Proof.* As before, let $k = \mathrm{rk}\, E$, $P \xrightarrow{\pi_0} M$ be the principal $G$-bundle associated to $E$, where $G \subset GL(k, \mathbb{C})$ is some complex Lie group, and $\alpha$ the connection form on $P$. $\alpha$ and $\pi_0$ induce, respectively, isomorphisms

$$\mathcal{V} \simeq P \times \mathfrak{g}$$

where $\mathcal{V} = \ker \mathrm{d}\pi_0$ and $\mathfrak{g}$ is the Lie algebra of $G$, and

$$\mathcal{H} \simeq \pi_0^* TM$$

where $\mathcal{H} = \ker \alpha$. Now take a holomorphic chart $z : U \to \mathbb{C}^n$ on an open set of $M$. We define an almost-complex structure on $P$ by taking as generators for the $(1,0)$-cotangents the components of $\alpha$, plus the components of $\mathrm{d}(z \circ \pi_0)$. More precisely we extend all these to $TP^c = TP \otimes \mathbb{C}$ and mark them to be $(1,0)$-forms. One can check this does not depend on the choice of the chart and thus $\mathrm{d}\pi_0$ becomes $\mathbb{C}$-linear. Since $R_g^*\alpha = \mathrm{Ad}\,(g^{-1})\alpha$ and $R_g^*\mathrm{d}(z\circ\pi_0) = \mathrm{d}(z\circ\pi_0\circ R_g) = \mathrm{d}(z\circ\pi_0)$, the right action of $G$ on $P$ is (pseudo-)holomorphic. We have thus defined an almost-complex structure on a space of orbits, namely

$$E = P\times_G \mathbb{C}^k$$

— and indeed on $P\times_G Y$ for every complex space $Y$ where $G$ acts holomorphically, and hence by biholomorphisms. Integrability of these almost-complex structures will arise from integrability of the almost-complex structure on $P$.

According to a version of the Newlander-Nirenberg's theorem (*cf.* [19]), the latter is integrable iff all the exterior derivatives of the $(1,0)$-forms have no $(0,2)$ part. Well, $\mathrm{dd}(z\circ\pi_0) = 0$. For $\alpha$, we start by choosing any section $s: U\rightarrow P$ (not necessarily holomorphic!) and write

$$\alpha = \mathrm{Ad}\,(\varphi^{-1})\pi_0^*\omega + \varphi^{-1}\mathrm{d}\varphi,$$

where $\varphi(sg) = g$, $\forall g\in G$, and $\nabla s = s\omega$ (*cf.* proposition 1.1). Now, to simplify computations, notice that $\mathrm{d}\alpha^{0,2} = 0$ iff $\mathrm{d}(\varphi\alpha)^{0,2} = 0$. Then

$$\begin{aligned}
\mathrm{d}(\varphi\alpha) &= \mathrm{d}(\pi_0^*\omega\varphi + \mathrm{d}\varphi)\\
&= \pi_0^*(\mathrm{d}\omega)\varphi - \pi_0^*\omega\wedge\mathrm{d}\varphi\\
&= \pi_0^*(\mathrm{d}\omega)\varphi - \pi_0^*\omega\wedge\varphi\alpha + \pi_0^*\omega\wedge\pi_0^*\omega\varphi\\
&= \pi_0^*\rho_U\varphi - \pi_0^*\omega\wedge\varphi\alpha,
\end{aligned}$$

where $\rho_U$ is the curvature form on $U\subset M$. Since $\pi_0^*$ preserves types we have $\mathrm{d}\alpha^{0,2} = 0$ iff $\rho_U^{0,2} = 0$.

One sees that a section $s = (s_1,\ldots,s_k)$ is holomorphic if $(s^*\alpha)'' = 0$, so $\nabla'' = \overline{\partial}_{\mathcal{E}}$. Finally, for each $\overline{\partial}_{\mathcal{E}}$ operator, its holomorphic structure — hence the uniqueness. ■

Notice that if $J$ is the almost-complex structure defined on $P$, then it preserves the splitting

$$TP = \mathcal{H}\oplus\mathcal{V}.$$

Indeed, for fundamental vertical vector fields, one can check that $J\tilde{A} = \widetilde{iA}$, so $\mathcal{V}$ and $\mathcal{H}$ become complex vector bundles. We can therefore ask when is the splitting holomorphic, and conjecture that it will be so iff $\alpha$ is holomorphic.

We know today that whatever might have appeared, to the reader, as original in the proof of the last theorem, then it is not: a remark in [3] already pointed to the ideas shown above. However, we wanted to understand the close relation between integrability on the bundle and the given condition on the base manifold.

Another interesting question is to find out which condition must a complex connection with $R^{0,2} = 0$ on a holomorphic vector bundle satisfy, in order to this holomorphic structure coincide with the one given by the theorem. The next result is proved in [16] but we give our own proof.

**Proposition 1.9.** *Let $D$ be a $\mathbb{C}$-linear connection on the tangent bundle of a complex manifold. Then $D'' = \overline{\partial}_{T_M}$ iff the torsion of $D$ is type $(2,0)$.*

*Proof.* One must start by noticing that the components of the $\mathbb{C}^n$-valued version of the form $\theta$ (see 1.9), *i.e.* when we identify $TM$ with $T'M = T^+M$ canonically, become holomorphic $(1,0)$-forms. In a chart $z$ of $M$, and with natural coordinates on the principal $GL(n,\mathbb{C})$-bundle of frames, and what is more identifying $T_zM = \mathbb{C}^n$, we have

$$\theta_{(z,g)} = g^{-1}\mathrm{d}z.$$

Now, if $D'' = \overline{\partial}$, then we can apply theorem 1.2. $\alpha$ becomes $(1,0)$ and holomorphic structures coincide, hence

$$\tau = \partial\,\theta + \alpha\wedge\theta$$

is $(2,0)$. If, reciprocally, $\tau$ is of this type, then $\alpha$ is $(1,0)$ because $\theta$ is non-degenerate on $\ker\alpha$, and therefore any section $s$ is holomorphic iff $(s^*\alpha)'' = 0$, so that $D'' = \overline{\partial}$. ∎

In the case of a linear connection, we thus have a possible answer to the question raised before.

In particular, we have the following straightforward corollary, for which a local correspondent on the base manifold is already known, *cf.* [16]. Note that it makes no sense at 'the unitary bundle level'.

**Corollary 1.1.** *(i) Any given hermitian connection over a hermitian manifold $(M, h)$ coincides with the unique $D$ defined in (1.10) if, and only if, its torsion is type* $(2, 0)$.

*(ii) $D$ is flat iff its associated connection 1-form is holomorphic.*

*Proof.* Recall that $\rho = \mathrm{d}\alpha + \alpha \wedge \alpha$ and that it is $(1, 1)$, so $\rho = \overline{\partial}\alpha$. ∎

Having studied the particular case of complex manifolds, we now proceed to take a close look at the relations between an almost-complex structure on a real manifold and a given linear connection. As usual, let $TM^c = TM \otimes \mathbb{C}$. In what follows, our notation will not distinguish between a real connection on $TM$ and its *twice* extension to an operator $\Gamma(TM^c) \to \Gamma(T^*M^c \otimes TM^c)$ — though not a connection, it satisfies (1.1) with complexified vector fields. On notation, the same ambiguity will hold with any covariant tensor. Instead, we denote with capital letters the real tangent vectors and with small letters their complexifications. Let $i = \sqrt{-1}$, let $\mathfrak{X}' = \Gamma T'M = \Gamma T^+M$ and $\mathfrak{X}'' = \overline{\mathfrak{X}'}$.

Let $\nabla$ be a linear connection on the almost-complex manifold $(M, J)$. We need two definitions from [22, 23]: $\nabla$ satisfies

$$\text{condition } (A) \text{ if} \qquad \nabla_{\overline{v}}\,\mathfrak{X}' \subset \mathfrak{X}', \qquad \forall v \in \mathfrak{X}', \tag{1.11}$$

and, $\nabla$ satisfies

$$\text{condition } (B) \text{ if} \qquad \nabla_{u}\,\mathfrak{X}' \subset \mathfrak{X}', \qquad \forall u \in \mathfrak{X}'. \tag{1.12}$$

Until the end of the section we show how the above conditions do interpret the relation between $J$, which is seen either as a section of $\mathrm{End}\,TM$ or as in $\mathrm{A}^1(TM)$, and the connection $\nabla$. Notice the latter is assumed to be torsion free on the second part of the following result.

**Proposition 1.10.** *Let $u, v \in \mathfrak{X}'$.*

*(i) We have always $(\nabla_X J)\,T'M \subset T''M$, for any $X \in \mathfrak{X}$.*

*(ii) Up to the equivalent conjugate of the above, other conditions like (A) and (B) are impossible to happen.*

*(iii) The following are equivalent to condition (A):*

$$(*)\ (\nabla_{\overline{u}}\, J)\, T'M = 0;$$

$$(**)\ J\nabla_X Y + J\nabla_{JX} JY = \nabla_X JY - \nabla_{JX} Y, \quad \forall X, Y \in \mathfrak{X}.$$

*Suppose now $\nabla$ is torsion free.*

*($\alpha$)* $\mathrm{d}^\nabla J\,(u,v) = i(1+iJ)[u,v] \in \mathfrak{X}''$;

*($\beta$)* $\mathrm{d}^\nabla J(u,\overline{v}) = -i(1-iJ)[u,\overline{v}] - 2i\nabla_{\overline{v}} u$*, hence*

*($\gamma$) condition (A) is* equivalent *to* $\mathrm{d}^\nabla J(u,\overline{v}) = 0$;

*($\delta$) condition (B) implies any of the following equivalent statements:*

$$\mathrm{d}^\nabla J(u,v) = 0 \qquad or \qquad [u,v] \in \mathfrak{X}' \qquad or \qquad J \ integrable.$$

*Proof.* *(i)* $J^2 = -\mathrm{Id}$, hence $(\nabla J)J + J(\nabla J) = 0$.

*(ii)* For example, if we had chosen $\nabla_u v \in \mathfrak{X}''$, then formula (1.1) would have led us to a contradiction.

*(iii)* For instance, like in [22], $(\nabla_{\overline{u}}\, J)\, v = \nabla_{\overline{u}}\,(J-i)\, v = -(J-i)\nabla_{\overline{u}}\, v$. *(**)* is just a restatement in real vectors.

*($\alpha$ and $\beta$)* The formula for $\mathrm{d}^\nabla$ still holds in $TM^c$, hence

$$\begin{aligned}
\mathrm{d}^\nabla J(u,\overline{v}) &= \nabla_u J\overline{v} - \nabla_{\overline{v}} Ju - J[u,\overline{v}] \\
&= -i\,(\nabla_{\overline{v}} u + [u,\overline{v}]) - i\nabla_{\overline{v}}\, u - J[u,\overline{v}] \\
&= -i(1-iJ)[u,\overline{v}] - 2i\nabla_{\overline{v}}\, u.
\end{aligned}$$

Even easier is the computation for *($\alpha$)*.

*($\gamma$ and $\delta$)* Recall $(1-iJ), (1+iJ)$ are projections to the $+$ and $-$ eigen-bundles, respectively. Then use the previous results. To get the first equivalence one also uses the following trick: *(A)* implies $\nabla_u \overline{v} \in \mathfrak{X}''$, by conjugation.

*($\delta$)* This becomes trivial to check; for example, condition *(B)* implies

$$[u,v] = \nabla_u v - \nabla_v u \in \mathfrak{X}'.$$

The last equivalence is the criteria of Newlander-Nirenberg for integrability of an almost-complex structure. ∎

Of course corresponding statements like *(iii)* above can be said for condition *(B)*. If the proposition already proves *(A)* and *(B)* to behave quite differently, the next result shows how they really correspond to different intrinsic properties of a riemannian manifold. It is a proposition of [23], thus a good chance to put our methods to the test.

**Proposition 1.11.** *For the Levi-Civita connection $\nabla^g$ on an almost-hermitian manifold $(M, h, J)$ we have*
*(i) condition (A) is satisfied iff* $\mathrm{d}\omega^{1,2} = 0$*, where* $\omega = g(J\ ,\ )$*;*
*(ii) condition (B) is also necessary for $J$ integrable.*

*Proof.* *(i)* Compatibility of $J$ with $g = \Re h$ assures

$$\omega = \frac{1}{2}\mathrm{Tr}\,(g\ J \wedge \mathrm{Id}),$$

as it is easy to see. Then, by propositions 1.8, 1.10 and because $g^{2,0} = 0$,

$$\begin{aligned}
\mathrm{d}\omega(u, v, \overline{w}) &= \frac{1}{2}\left\{g(\mathrm{d}^\nabla J(u, v), \overline{w}) + g(\mathrm{d}^\nabla J(v, \overline{w}), u) + g(\mathrm{d}^\nabla J(\overline{w}, u), v)\right\} \\
&= \frac{1}{2}\left\{-2ig(\nabla_{\overline{w}}\, v, u) - g(\mathrm{d}^\nabla J(u, \overline{w}), v)\right\} \\
&= i(g(\nabla_{\overline{w}} u, v) + g(\nabla_{\overline{w}} u, v)) \\
&= 2ig(\nabla_{\overline{w}} u, v).
\end{aligned}$$

*(ii)* Let $A(u, v, w) = g(\nabla_u v, w)$ — a 3-tensor on $T'M$ since $g^{2,0} = 0$. Then

$$g(\nabla_u v, w) = u \cdot g(v, w) - g(v, \nabla_u w) = -g(v, \nabla_u w)$$

and

$$0 = g([u, v], w) = g(\nabla_u v, w) - g(\nabla_v u, w),$$

assuming $J$ is integrable. Hence $A$ is skew in $(v, w)$ and symmetric in $(u, v)$, which implies $A = 0$, and thus $\nabla_u v \in \mathfrak{X}'$. Notice $g^c$ is non-degenerate. ■

If $\mathrm{d}\omega^{1,2} = 0$, then $M$ is called $(1, 2)$-symplectic. Finally we get the following famous result, which leads us back to the starting point of view.

**Theorem 1.3.** *Let $(M, h, J)$ be almost-hermitian. Any of the following serve as definition of a Kähler manifold:*

*(i) $M$ admits a hermitian, torsion free connection;*

*(ii) $\nabla^g J = 0$;*

*(iii) $J$ is integrable and $M$ is $(1,2)$-symplectic (hence symplectic).*

*Proof.* Besides $h = g + i\omega$, notice all objects are real, and thus $Dh = 0$ iff $Dg = 0$. This proves *(i)* iff *(ii)*. For the rest, in any case $\omega$ is type $(1,1)$ since $g$ is. Clearly *(ii)* is the same as *(A)+(B)*. Then recall that only if $J$ is integrable we can say $\mathrm{d}\omega^{3,0} = \mathrm{d}\omega^{0,3} = 0$. ∎

It is intriguing that $\mathrm{d}\omega^{3,0} = 0$ does not imply integrability. However, the latter arising from the vanishing of just one $\mathbb{C}$-valued form would be surprising. We can ask, then, if it is possible to find a manifold for which only $\mathrm{d}\omega^{3,0} = 0$.

Notice that there are both examples of almost-hermitian manifolds satisfying one of the conditions *(A)* or *(B)* and not the other. For *(A)*, non-kählerian symplectic manifolds are famous by now and, more refined, $S^6$ admits a $(1,2)$-symplectic structure according to [23]. It is not known to have an integrable complex structure. For *(B)*, we have the many hermitian non-kählerian spaces.

## 1.4 Symplectic connections

Let $M, N$ be two given manifolds and $\sigma : M \to N$ a diffeomorphism between them. Let $\nabla$ be a linear connection on $M$. Then recall that we can define another connection on $N$ by

$$(\sigma \cdot \nabla)_X Y = \sigma \cdot \left(\nabla_{\sigma^{-1}\cdot X}\sigma^{-1} \cdot Y\right), \tag{1.13}$$

where $X, Y \in \mathfrak{X}_N$ and

$$\sigma \cdot Z_y = \mathrm{d}\sigma(Z_{\sigma^{-1}(y)})$$

for any $Z \in \mathfrak{X}_M,\ y \in N$. This connection will appear many times in this work — also as object of our study. It is well defined, at least, on paracompact manifolds.

Indeed, from any tensor on $M$ we can define another one on $N$. Notice as well that $\sigma\cdot fZ = (f\circ\sigma^{-1})\sigma\cdot Z = \sigma\cdot f\ \sigma\cdot Z$, for all $f\in \mathrm{C}^\infty_M$, so we prove the last statement and check (1.1) for $\sigma\cdot\nabla$. Furthermore

$$T^{\sigma\cdot\nabla} = \sigma\cdot T^\nabla, \qquad R^{\sigma\cdot\nabla} = \sigma\cdot R^\nabla,$$

since $\sigma\cdot[Z,W] = [\sigma\cdot Z, \sigma\cdot W]$. Obvious composition rules are satisfied and

$$(\sigma\cdot\nabla)_X\omega = \sigma\cdot\left(\nabla_{\sigma^{-1}\cdot X}\sigma^*\omega\right)$$

for any form $\omega$ on $N$. For instance, let us prove the last formula:

$$\begin{aligned}
\sigma\cdot\left(\nabla_{\sigma^{-1}\cdot X}\sigma^*\omega\right)(\ldots,Y_i,\ldots) &= \left(\nabla_{\sigma^{-1}\cdot X}\sigma^*\omega\right)(\ldots,\sigma^{-1}\cdot Y_i,\ldots) \\
&= (\sigma^{-1}\cdot X)\left(\sigma^*\omega(\sigma^{-1}\cdot Y_1,\ldots,\sigma^{-1}\cdot Y_q)\right) \\
&\qquad -\sum_i \sigma^*\omega\left(\sigma^{-1}\cdot Y_1,\ldots,\nabla_{\sigma^{-1}\cdot X}\sigma^{-1}\cdot Y_i,\ldots,\sigma^{-1}\cdot Y_q\right) \\
&= \mathrm{d}(\omega_\sigma(Y_1,\ldots,Y_q))(\mathrm{d}\sigma^{-1}(X)) - \sum\omega\left(Y_1,\ldots,(\sigma\cdot\nabla)_X Y_i,\ldots,Y_q\right) \\
&= (\sigma\cdot\nabla)_X\,\omega(\ldots,Y_i,\ldots).
\end{aligned}$$

As we said above, $\sigma^{-1}\cdot\omega = \sigma^*\omega$.

**Remark.** In a marginal outlook to this theory we find that all *characteristic classes* on $TM$, constructed like in proposition 1.4 with the multilinear forms $f^i$ say $G$-invariant, are fixed points of cohomology for every diffeomorphism preserving some $G$-structure of $M$. And the proof goes as follows. Taking any $G$-connection $\nabla$, assumed to exist, we then have

$$\begin{aligned}
f^i(R^{\sigma^{-1}\cdot\nabla},\ldots,R^{\sigma^{-1}\cdot\nabla}) &= f^i(\mathrm{d}\sigma^{-1}(\sigma^*R^\nabla)\mathrm{d}\sigma,\ \ldots\ ) \\
&= \sigma^* f^i(R^\nabla,\ldots,R^\nabla).
\end{aligned}$$

Hence, by the independence of the induced de Rham cohomology classes from the connection, the former are fixed points for $\sigma^*$. Of course the result is interesting only when $Diff(M)$ has many arcwise-connected components.

From now on we are interested in the case where $M$ and $N$ are symplectic manifolds and $\sigma$ is a symplectomorphism. A linear connection on $(M,\omega)$ is called symplectic if $\nabla\omega = 0$ *and* if it is torsion free. In such case, we have that $\sigma \cdot \nabla$ is symplectic too. Indeed, this follows easily from the listed formulae. In particular we have an action

$$Symp(M,\omega) \times \mathcal{A} \longrightarrow \mathcal{A}$$

on the space of symplectic connections, which preserves the subspace of flat connections. $\mathcal{A}$ is never empty.

**Theorem 1.4.** *Every symplectic manifold admits a symplectic connection.*

We just present the formulas from the proof, which is due to P. Tondeur (see [6]): starting with any connection $\nabla$, adding $-\frac{1}{2}T^{\nabla}$ if necessary, we may assume it to be torsion free. Then

$$\omega(\nabla^1_X Y, Z) = \omega(\nabla_X Y, Z) + \frac{1}{3}(\nabla_X \omega)(Y,Z) + \frac{1}{3}(\nabla_Y \omega)(X,Z)$$

defines a symplectic connection $\nabla^1$. Moreover, if a Lie group $H$ acts on $M$ by symplectomorphisms and thus on the space of connections, then $M$ has a $H$-invariant connection if and only if it has a $H$-invariant symplectic connection, that is, a fixed point in $\mathcal{A}$.

Notice that a manifold with a non-degenerate 2-form and a symplectic torsion free connection is necessarily symplectic, by proposition 1.8.

We now make an overview of some recent results from [6, 8, 9, 10, 25]. Though we will not use them much more they constitute an important part of the theory of symplectic connections.

In order to choose a smaller subspace of $\mathcal{A}$ it was introduced a variational principle

$$\int R^2\,\omega^n,$$

which can be interpreted as an $L^2$-scalar product in curvature like tensors. Notice $2n$ is the dimension of $M$. First one defines the tensor

$$\underline{R}(X,Y,Z,T)=\omega(R(X,Y)Z,T),$$

which verifies *the* first Bianchi identity (recall $T^\nabla=0$) and a second Bianchi identity

$$\underset{X,Y,Z}{\circlearrowright\!\!\!\!\!\bigoplus}\,(\nabla_X\underline{R})(Y,Z,T,U)=0,$$

and then the Ricci tensor $r(X,Y)=\mathrm{Tr}\,\{Z\mapsto R(X,Z)Y\}$. Under the action of $Sp(2n,\mathbb{R})$ on the space of tensors like $\underline{R}$ the representation theory has been done, and thus it is known that the curvature of $\nabla$ has two irreducible components — a very important part of the theory which is due to I. Vaisman ([25]). So we write $\underline{R}=E+W$ where

$$\begin{aligned}E(X,Y,Z,T)=-\frac{1}{2(n+1)}\Big\{&2\omega(X,Y)r(Z,T)+\omega(X,Z)r(Y,T)\\&+\omega(X,T)r(Y,Z)-\omega(Y,Z)r(X,T)-\omega(Y,T)r(X,Z)\Big\}.\end{aligned}$$

$R^2$ is defined as the full contraction of $\underline{R}$ with the tensor $\omega$ and must agree with $E^2+W^2$.

The connection is said to be of Ricci type if $W=0$, and this Weyl part $W$ is a symplectic equivalent to the Weyl curvature tensor in conformal riemannian geometry. In our case too, it is 0 in dimension 2. The variational principle yields the field equations

$$\underset{X,Y,Z}{\circlearrowright\!\!\!\!\!\bigoplus}\,(\nabla_X r)(Y,Z)=0, \tag{1.14}$$

having as particular solutions the Ricci type connections.

**Theorem 1.5.** *Let $(M,\omega)$ be a symplectic manifold with a Ricci type symplectic connection. Then there exists a 1-form $u$ such that*

$$(\nabla_X r)(Y,Z) = \omega(X,Y)u(Z) + \omega(X,Z)u(Y).$$

*Conversely, if there is such a 1-form $u$, the Weyl part of the curvature, $W = \underline{R} - E$, satisfies*

$$\circlearrowright\!\!\!\!\!\!\bigoplus_{X,Y,Z} (\nabla_X W)(Y,Z,T,U) = 0.$$

For the proof see [10]. A remarkable discovery in [10] is that, if $(M_i, \omega_i, \nabla_i)$ are symplectic manifolds together with corresponding symplectic connections and such that the symplectic $\nabla = \nabla_1 + \nabla_2$ over the cartesian product $(M_1 \times M_2, \omega_1 + \omega_2)$ is of Ricci type, then all three connections must be flat.

A result proved in [8] shows that with its standard metric $P^n(\mathbb{C})$ is of Ricci type — as it is implicit in the last section, the Levi-Civita connection becomes a symplectic connection in the kählerian framework.

As an example here and for later purposes we show the following result. Consider $(\mathbb{R}^2, \omega)$ with its standard symplectic structure

$$\omega = \frac{i}{2} \mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

Let $z = x + iy$ represent the usual coordinates, so that $\omega = \mathrm{d}x \wedge \mathrm{d}y$, and let

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

**Proposition 1.12.** *Every symplectic connection on $(\mathbb{R}^2, \omega)$ is uniquely determined by two functions $\alpha, \beta \in \mathrm{C}^\infty_M(\mathbb{C})$ satisfying*

$$\nabla_{\partial_z} \partial_z = \alpha\, \partial_z + \beta\, \partial_{\overline{z}} = \overline{\nabla_{\partial_{\overline{z}}} \partial_{\overline{z}}}$$

*and*

$$\nabla_{\partial_z} \partial_{\overline{z}} = -\overline{\alpha}\, \partial_z - \alpha\, \partial_{\overline{z}} = \nabla_{\partial_{\overline{z}}} \partial_z.$$

*The flat symplectic connections are given by the system*

$$\begin{cases} \frac{\partial \alpha}{\partial \overline{z}} + \frac{\partial \overline{\alpha}}{\partial z} + |\beta|^2 - |\alpha|^2 = 0 \\[2em] \frac{\partial \beta}{\partial \overline{z}} + \frac{\partial \alpha}{\partial z} - 2\alpha^2 + 2\overline{\alpha}\beta = 0. \end{cases}$$

The proof is elementary. Indeed, the *real* and torsion free assumptions, together with

$$\frac{i}{2}\alpha = \omega(\nabla_{\partial_z}\partial_z, \partial_{\overline{z}}) = -\omega(\partial_z, \nabla_{\partial_z}\partial_{\overline{z}}),$$

show us the first part. Notice that for the second part we just have to develop one equation:

$$\nabla_{\partial_z}\nabla_{\partial_{\overline{z}}}\partial_z - \nabla_{\partial_{\overline{z}}}\nabla_{\partial_z}\partial_z = 0,$$

but the result will not be used anymore. Because sometimes is impossible to *keep* complex, we give the real correspondent of the last proposition. If

$$\begin{aligned} \nabla_{\partial_x}\partial_x &= b\,\partial_x - a\,\partial_y, & \nabla_{\partial_y}\partial_y &= d\,\partial_x - c\,\partial_y, \\ \nabla_{\partial_x}\partial_y &= c\,\partial_x - b\,\partial_y = \nabla_{\partial_y}\partial_x, \end{aligned}$$

then

$$\alpha = -\frac{b+d}{4} - i\frac{a+c}{4} \qquad \text{and} \qquad \beta = \frac{3b-d}{4} - i\frac{3c-a}{4}.$$

There is a striking and long standing question about connections on $\mathbb{R}^m$ for which we do not have an answer, but would like to help to clarify. To start with we shall restrict to the case $m = 2$ and present a result which can be easily generalised. Let $s$ denote the global frame

$$s = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

and let $\nabla^0 = \mathrm{d}$ be the trivial connection: in the example above, $\alpha = \beta = 0$.

**Proposition 1.13.** *Suppose $\nabla = \sigma \cdot \nabla^0$, for some $\sigma \in Symp(\mathbb{R}^2, \omega)$. Then*

$$\nabla s = sg\mathrm{d}g^{-1}$$

*where $g \circ \sigma = \mathrm{Jac}\,\sigma$.*

*Proof.* Consider the theory in pages 7 and 8. With $s_{_V} = s$ and $s_{_U} = \sigma \cdot s$ we have

$$\nabla s_{_U} = \sigma \cdot \left(\nabla^0_{\sigma^{-1}\cdot}\, s\right) = 0,$$

thus the respective connection 1-forms verify $\omega_U = 0$ and $\omega_V = g\mathrm{d}g^{-1}$, with $g$ given by $s_V = s_U g^{-1}$ or simply $sg = \sigma\cdot s$. Since for $\sigma = (\sigma_1, \sigma_2)$

$$\begin{aligned} \sigma\cdot s &= \left(\mathrm{d}\sigma\Big(\frac{\partial}{\partial x}{}_{\sigma^{-1}}\Big), \mathrm{d}\sigma\Big(\frac{\partial}{\partial y}{}_{\sigma^{-1}}\Big)\right) \\ &= \left(\frac{\partial\sigma_1}{\partial x}{}_{(\sigma^{-1})}\frac{\partial}{\partial x} + \frac{\partial\sigma_2}{\partial x}{}_{(\sigma^{-1})}\frac{\partial}{\partial y}, \ \ldots\right) \\ &= s\mathrm{Jac}\,\sigma_{|\sigma^{-1}} \end{aligned}$$

we may conclude the result. ■

The question is: what is the orbit of $\nabla^0$ under the symplectomorphisms' group action? Necessary conditions for $\nabla$ to be in that orbit are that it must be flat and torsion free — for instance $R^\nabla = \sigma\cdot R^{\nabla^0} = 0$ — but it is not known if the former are sufficient.

The system of partial differential equations

$$\nabla = \sigma\cdot\nabla^0$$

in variable $\sigma$ is $2^{\mathrm{nd}}$-order nonlinear, but we can check that it is a $1^{\mathrm{st}}$-order linear composed with $\mathrm{Jac}\,\sigma_{|\sigma^{-1}}$. Moreover, $R^\nabla = 0$ translates as the integrability condition of the latter linear equation. If one does not want to recur to the theory of differential equations to see this, then we can rely on the theory of connections and recall that there exist local parallel frame fields for $\nabla$. Then, translating back all the parallel frames $s_i$ on the cover $\{U_i\}$ of $\mathbb{R}^2$ to the frame $s$ by $s_i = sg_i$, we have that $g_i g_j^{-1}$ is constant. So it is a very simple problem of the cohomological type to construct a global (not unique) map $g : \mathbb{R}^2 \to SL(2) = Sp(2,\mathbb{R})$ such that $\nabla s = sg\mathrm{d}g^{-1}$. Still, do there exist $\sigma \in Symp(\mathbb{R}^2, \omega)$ such that

$$g\circ\sigma = \mathrm{Jac}\,\sigma \qquad ?$$

This time $T^\nabla = 0$ is the necessary condition for the last equation to have solutions, namely Schwartz equality of crossed derivatives.

Supposing solutions $\sigma$ exist, composing them with any translation $x \mapsto x + v,\ v \in \mathbb{R}^2$, will also give a solution. Indeed, these maps are in the isotropy subgroup of $\nabla^0$, which by proposition 1.13 is $Symp(\mathbb{R}^2,\omega)_{\nabla^0} = SL(2) \rtimes \mathbb{R}^2$. So we may look for $\sigma$ such that $\sigma(0) = 0$. Moreover, assuming $g(0) = 1$ is not a problem either, as one deduces from the formula in the proposition — it corresponds henceforth to a gauge transformation.

To illustrate the question above and others to follow, within the framework of symplectic geometry, we give a simple (!) example: consider the open set $\mathbb{R}^+ \times \mathbb{R}$ and, as in the previous example, in real coordinate functions, take the connection $a = c = 0,\ \ d = x$ and $b = -\frac{1}{2x}$. An easy computation shows $\nabla$ is flat. A little extra work to find the group-valued map $g$, leads then to the problem of finding $(\sigma_1, \sigma_2)$ such that

$$\begin{bmatrix} \frac{\sqrt{2\sigma_1}}{2} e^{-\frac{\sigma_2}{\sqrt{2}}} & -\sqrt{\sigma_1} e^{\frac{\sigma_2}{\sqrt{2}}} \\ \frac{1}{2\sqrt{\sigma_1}} e^{-\frac{\sigma_2}{\sqrt{2}}} & \frac{\sqrt{2}}{2\sqrt{\sigma_1}} e^{\frac{\sigma_2}{\sqrt{2}}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \sigma_1}{\partial x} & \frac{\partial \sigma_1}{\partial y} \\ \frac{\partial \sigma_2}{\partial x} & \frac{\partial \sigma_2}{\partial y} \end{bmatrix}$$

Finally, can one solve the equations for any *rational* connection? What has Poisson geometry to say besides $\{\sigma_1, \sigma_2\} = \det \operatorname{Jac} \sigma = 1$? Demanding $\nabla$ to be complete, *i.e.* all parallel curves along themselves can be defined *ad infinitum*, what difference would it make? What are the orbits and isotropy subgroups of any other connection?

There is a type of connection for which we have found a solution to the problem raised before. Consider a symplectic connection in $\mathbb{R}^{2n}$ which is translation invariant, that is $T_v \cdot \nabla = \nabla$ for all maps $T_v(x) = x + v,\ v \in \mathbb{R}^{2n}$. Letting $\nabla = \nabla^0 + A$ where $A$ is a $\mathfrak{sp}(2n, \mathbb{R})$-valued 1-form, we thus must have

$$T_v \cdot \nabla = \nabla^0 + T_v \cdot A = \nabla^0 + A.$$

Since $\mathrm{d}T_v = \mathrm{Id}$, one does not take long to conclude that $A_{x+v} = A_x$, *i.e.* $A$ is a constant 1-form.

The following theorem has appeared in [9] proved with entirely different methods.

**Theorem 1.6.** *Let $\nabla$ be a flat, translation invariant and symplectic connection on the manifold $\mathbb{R}^{2n}$. Suppose $\nabla = \nabla^0 + A$. Then $A(X)A(Y) = 0$ for all vectors*

*$X, Y$, and with the map*

$$\sigma(x) = x - \frac{1}{2}A(x)x$$

*we have $\nabla = \sigma \cdot \nabla^0$.*

*Proof.* By a formula justified in proposition 1.4 we get

$$\begin{aligned} 0 = R^{\nabla} &= R^{\nabla^0} + \mathrm{d}^{\nabla^0} A + A \wedge A \\ &= A \wedge A \end{aligned}$$

so that $[A(X), A(Y)] = 0$. Hence, to see $A(X)A(Y) = 0$ we just have to show $A(X)A(X) = 0$. Let $X \in \mathbb{R}^{2n}$ be fixed and consider the 2-form

$$\alpha(Y, Z) = \omega(A(X)Y, A(X)Z).$$

By the torsion free assumption, $A(X)Y = A(Y)X$, so $\alpha$ also satisfies

$$\begin{aligned} \alpha(Y, Z) &= \omega(A(Y)X, A(Z)X) \\ &= -\omega(A(Y)A(Z)X, X) \end{aligned}$$

and hence, being symmetric, it must vanish — which implies $A(X)A(X) = 0$. This proves the first part of the theorem.

From the results after proposition 1.13 we have seen that $A = g\mathrm{d}g^{-1}$ for some global $g \in \mathrm{A}^0(Sp(2n, \mathbb{R}))$. Let us find $g$, which will thus determine the connection completely. Certainly, in canonical coordinates $(x_1, \ldots, x_{2n})$

$$A = \sum A_i \mathrm{d}x_i = \mathrm{d}\left(\sum x_i A_i\right)$$

with constant $A_i$. Now let $B = \sum x_i A_i = A(x)$. Again, $\mathrm{d}B\,B = B\mathrm{d}B$ so it is trivial to check that

$$g = e^{-B} = \sum \frac{(-B)^m}{m!}.$$

According to the same proposition 1.13 we are left to solve the equation

$$e^{-A(\sigma)} = \mathrm{Jac}\,\sigma$$

or equivalently

$$1 - A(\sigma) = \mathrm{Jac}\,\sigma.$$

In the canonical basis $(e_i)$ of $\mathbb{R}^{2n}$, this is the same as

$$e_i - A(\sigma)e_i = \frac{\partial\sigma}{\partial x}^i.$$

Letting $\sigma(x) = x - \frac{1}{2}A(x)x$ then we have that

$$\begin{aligned} \frac{\partial\sigma}{\partial x}^i &= e_i - \frac{1}{2}A(e_i)x - \frac{1}{2}A(x)e_i \\ &= e_i - A(x)e_i \end{aligned}$$

and that

$$A(\sigma(x)) = A(x) - \frac{1}{2}A((A(x)x) = A(x),$$

so the given map satisfies the differential equation, as we wished. ■

We acknowledge the help of [9] in seeing the $A(X)A(Y) = 0$ part, in dimensions $2n \geq 4$.

One may easily find the set of non-zero 1-forms $A$ representing flat, translation invariant symplectic connections in $\mathbb{R}^2$, up to a scalar factor. It is in 1-1 correspondence with the *non-empty* curve

$$\Big\{[a:b:c:d] \in P^3(\mathbb{R}) :\ bc - ad = 0,\ b^2 - ac = 0\Big\}\backslash\{pt\}$$

where $pt = [0:0:1:0]$.

# Chapter 2

## 2.1 The fibre of the twistor space

In the beginning of the first two sections of this chapter we intend to follow very closely the theory of twistor spaces as presented in [20]. The twistor space is a bundle over some manifold with a particular fibre which we start by recalling.

Let $V$ be a fixed real vector space of even dimension $2n$. Consider

$$\mathcal{J}(V)=\left\{J\in\operatorname{End}V:\ J^2=-1\right\},$$

*i.e.* the space of all complex structures in $V$. For each of its elements there are two associated complex $n$-dimensional subspaces of $V^c$: the $+i$ and $-i$ eigenspaces of say $J$, which we denote respectively by $V^+$ and $V^-$. Choosing, by induction, a basis of $V^+$ of the kind $\{X_m-iJX_m\}_{m=1,\dots,n}$, then $\{X_m,JX_m\}$ becomes a real basis of $V$, and *vice-versa*. We recall that the conjugation map is a natural involution of $V^c$ which implies the equal $\mathbb{C}$-dimensions of those eigenspaces and hence the requirement of even real dimension.

Consider the action of the real $GL(V)$ on $\mathcal{J}(V)$

$$J\longmapsto gJg^{-1}.$$

From the above we see that this action is transitive and thus that

$$\mathcal{J}(V)=\frac{GL(V)}{GL(V,J)}.$$

As a homogeneous space, the tangent space to $\mathcal{J}(V)$ at $J$ is identified with

$$\mathfrak{m}_{_J}=\{A\in\mathfrak{gl}:\ AJ=-JA\}.$$

Indeed, a decomposition of any $A \in \mathfrak{gl}$ as $2A = A + JAJ + A - JAJ$ shows $\mathfrak{m}_J$ to be a complement of $\mathfrak{gl}(V, J) = T_J GL(V, J)$ in the whole $\mathfrak{gl}$. It is easy to check the relations

$$[\mathfrak{gl}(V,J), \mathfrak{gl}(V,J)] \subset \mathfrak{gl}(V,J), \qquad [\mathfrak{gl}(V,J), \mathfrak{m}_J] \subset \mathfrak{m}_J,$$

$$[\mathfrak{m}_J, \mathfrak{m}_J] \subset \mathfrak{gl}(V,J),$$

so $\mathcal{J}(V)$ is a symmetric space. Notice $JAJ = -JJA$ for $A \in \mathfrak{m}_J$, so left multiplication by $J$ induces a linear endomorphism $\beta_J$ of $\mathfrak{m}_J$ such that

$$\beta_J^2 = -\mathrm{Id},$$

*i.e.* an almost-complex structure on $\mathcal{J}(V)$.

Considering again the complex reflection of all this, let $J^+, J^-$ be the projections

$$J^+ = \frac{1}{2}(1 - iJ), \qquad J^- = \frac{1}{2}(1 + iJ)$$

of $V^c$ to $V^+$ and $V^-$ respectively, so that $1 = J^+ + J^-$ and $J^+J^- = J^-J^+ = 0$. Also we have a direct sum decomposition

$$\mathfrak{gl}(V)^c = \mathfrak{gl}(V,J)^c + \mathfrak{m}_J^+ + \mathfrak{m}_J^- \tag{2.1}$$

into the $0, +i, -i$ eigenspaces of $\frac{1}{2}\mathrm{ad}\, J$ — this agrees with $\beta_J$ on $\mathfrak{m}_J$. Thus

$$[\mathfrak{gl}(V,J)^c, \mathfrak{m}_J^\pm] \subset \mathfrak{m}_J^\pm, \qquad [\mathfrak{m}_J^+, \mathfrak{m}_J^+] = 0$$

and

$$[\mathfrak{m}_J^+, \mathfrak{m}_J^-] \subset \mathfrak{gl}(V,J)^c.$$

For example, $[A - iJA, B - iJB] = [A, B] - [JA, JB] - i([JA, B] + [A, JB]) = 0$, for all $A, B \in \mathfrak{m}_J$. Henceforth, according to [17, 20], $\mathcal{J}(V)$ has a unique *real* $GL(V)$-invariant complex structure whose $(1,0)$-tangent space at $J$ is $\mathfrak{m}_J^+$.

**Proposition 2.1.** *If* $a = a^0 + a^+ + a^- \in \mathfrak{gl}(V)^c$ *denotes the decomposition of* $a$ *with respect to (2.1) then*

$$a^0 = J^+aJ^+ + J^-aJ^-, \qquad a^+ = J^+aJ^- \quad \textit{and} \quad a^- = J^-aJ^+.$$

*Proof.* $J^+ + J^- = 1$, so $a = J^+aJ^+ + J^+aJ^- + J^-aJ^+ + J^-aJ^-$. Then

$$[J, a^0] = iJ^+aJ^+ - iJ^-aJ^- - iJ^+aJ^+ + iJ^-aJ^- = 0,$$

since $J$ on $V^c$ is $(J, J)$ on $V^+ \oplus V^-$. The rest is proved just the same way. ■

The proposition can be found in [20].

We now specialise to a subspace of $\mathcal{J}(V)$. Suppose $\omega$ is a symplectic form on the real vector space $V$. Let

$$J(V, \omega, *) = \left\{ J \in \mathcal{J}(V) : \ \omega = \omega^{1,1} \ \text{for } J \right\}.$$

Then for any vectors $X, Y$

$$\begin{aligned} 0 &= \omega(X - iJX, Y - iJY) \\ &= \omega(X, Y) - \omega(JX, JY) - i(\omega(X, JY) + \omega(JX, Y)), \end{aligned}$$

so the new imposed condition is the same as $J$ being a symplectic linear transformation of $V$, or what is called 'compatible' with $\omega$. Consider the symmetric form

$$g_J = \omega(\ , J\ ).$$

This non-degenerate inner product has *even* signature $(2n - 2l, 2l)$, for some $0 \leq l \leq n$, because any maximal subspace where it is positive definite is $J$-invariant. We denote by $J(V, \omega, l)$ the $n+1$ connected components, as we shall see according to the index $2l$, of the disjoint union $J(V, \omega, *)$.

**Remark** on the sign rule: when $l = 0$ we want $-i\omega(u, \overline{u}) = g_J(u, \overline{u}) \geq 0$. This gives the usual hermitian structure in $V^+$. Also, without loss of generality assume $n = 1$, we have that $-i\omega(\partial_z, \partial_{\overline{z}}) = \frac{1}{2}$ in canonical $\mathbb{R}^2$ (*cf.* section 1.4), and this corresponds to the complex structure

$$J_0 \left( \frac{\partial}{\partial x} \right) = \frac{\partial}{\partial y}.$$

Finally, in oriented basis $\{\partial_x, \partial_y\}$ and in matrix form, $\omega(X,Y) = -X^t J_0 Y$.

The following generalisation of a lemma from [18] explains the topological remark above. We give a different proof in appendix A.

**Proposition 2.2.** *(i) None of the $J(V,\omega,l)$ are empty.*
*(ii) The action (2.1) of $Sp(V,\omega)$ on $J(V,\omega,l)$ is transitive.*

Thus

$$J(V,\omega,l) \;=\; \frac{Sp(V,\omega)}{U(n-l,l)},$$

a quotient by the pseudo-unitary group. Fixing any compatible $J$ with index $2l$, we have an inner automorphism $g \mapsto -JgJ$ of $GL(V)$ which preserves $Sp(V,\omega)$ and the respective $U(n-l,l)$. Appealing to the theory we observe that the subspaces we have just been describing are *symmetric-subspaces* of $\mathcal{J}(V)$ (*cf.* [17]). Clearly $J \in \mathfrak{u}(n-l,l)$ too, so there is a direct sum

$$\mathfrak{sp}(V,\omega) = \mathfrak{u}(n-l,l) + \mathfrak{n}_{_J}$$

where $\mathfrak{n}_{_J} = \mathfrak{sp}(V,\omega) \cap \mathfrak{m}_{_J}$. One easily checks that $\mathfrak{n}_{_J}$ is preserved under left multiplication by $J$, thus the $J(V,\omega,l)$ are also complex submanifolds of $\mathcal{J}(V)$.

To make a short break, we recall the 'Siegel upper half space' or 'Siegel domain'

$$\mathcal{D}_n = \left\{ z \in \mathbb{C}^{\frac{1}{2}n(n+1)} : \; z \text{ symmetric}, \; \Im z \text{ positive definite} \right\}$$

where the elements $z$ are $n \times n$ matrices with complex entries. $G = Sp(2n,\mathbb{R})$ acts transitively on $\mathcal{D}_n$ by

$$(g,z) \longmapsto g \cdot z = (az+b)(cz+d)^{-1}, \qquad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G,$$

where $a,b,c,d$ are square matrices. To see that this is well defined and the action is transitive we appeal to [24]. Easy enough is that the stabiliser of $i1$ is the subgroup of those $g$ for which $d = a$ and $c = -b$, that is the most convenient $U(n)$. Hence we recover $J(V, \omega, 0)$.

Now we look at $\mathcal{D}_n$ as an open complex manifold.

**Proposition 2.3.** *The map* $\phi : \mathcal{D}_n \to J(V, \omega, 0)$ *given by*

$$x + iy \longmapsto \begin{bmatrix} xy^{-1} & -xy^{-1}x - y \\ y^{-1} & -y^{-1}x \end{bmatrix}$$

*is a* $G$*-equivariant* anti-*biholomorphism.*

*Proof.* Of course $\phi(i1) = J_0$. Suppose $g \in G$ is such that

$$g \cdot i1 = x + iy.$$

Then $\phi(x + iy)$ must be equal to $gJ_0g^{-1}$. Using the well known relations between the four squares inside $g$, which also give $g^{-1}$, we can use the equation above to write $gJ_0g^{-1}$ in terms of $x$ and $y$. Or rather one can check directly that the matrix presented is a true element of $J(\mathbb{R}^{2n}, \omega, 0)$. By construction, $\phi$ is $G$-equivariant.

Since $G$ acts by rational maps in variable $z$, hence by holomorphic transformations of $\mathcal{D}_n$, we may conclude that multiplication by $i$ in $T\mathcal{D}_n$ agrees with a $G$-invariant complex structure. It remains to show that this is the same as *right* multiplication by $J_0$ in $\mathfrak{n}_{J_0}$, up to the isomorphism

$$\mathrm{d}p_| : \mathfrak{n}_{J_0} \longrightarrow T_{i1}\mathcal{D}_n$$

arising from the projection $p : G \to \mathcal{D}_n,\ g \mapsto g \cdot i1$. If we denote

$$E = \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} \qquad \text{and} \qquad g(t) = \exp(tE) = \begin{bmatrix} a_t & b_t \\ c_t & d_t \end{bmatrix},$$

then the following derivative taking place at point 1 makes sense.

$$\begin{aligned} \mathrm{d}p(E) &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}(a_ti + b_t)(c_ti + d_t)^{-1} \\ &= \dot{a}_0 i + \dot{b}_0 - i(\dot{c}_0 i + \dot{d}_0) = (e_1 - e_4)i + e_2 + e_3. \end{aligned}$$

Now the result

$$\mathrm{d}p(J_0E) = (-e_3 - e_2)i + e_1 - e_4 = -i\,\mathrm{d}p(E)$$

is immediate to check. ∎

We are aware that these structures are canonical so the $\mathbb{C}$-analycity was likely to correspond. Also notice that simple counter-examples show the obvious generalisation of the above to any signature not to hold consistent. However, we will call equally 'Siegel domain' to all $J(V,\omega,l)$.

After this digression let us now search for a compact symmetric space in which to embed all the $J(V,\omega,l)$. We appeal to the theory of flag manifolds and parabolic subgroups. As in [7], let $G^{^C}$ be a connected semisimple complex Lie group and $\mathfrak{g}^C$ its Lie algebra. As we saw in [7], a parabolic subgroup $P$ is the normaliser in $G^{^C}$ of a parabolic subalgebra of $\mathfrak{g}^{^C}$, this is, a complex Lie subalgebra $\mathfrak{p}$ which contains a maximal solvable subalgebra of $\mathfrak{g}^{^C}$, or a Borel subalgebra. We recall,

$$P = \left\{ g \in G^{^C} \;:\; \mathrm{Ad}\,(g)\mathfrak{p} \subset \mathfrak{p} \right\}.$$

The theory says that $\mathrm{Lie}(P) = \mathfrak{p}$ and that, if $K$ is a compact real form of $G^{^C}$, then

$$F = \frac{G^{^C}}{P} \simeq \frac{K}{K \cap P}$$

because $K$ acts transitively on $G^{^C}/P$. These compact and complex spaces are the so called flag manifolds. If $G$ is a non-compact real form of $G^{^C}$ then it may not act transitively on $F$. The open orbits of this action are called flag domains.

Applying this theory to $Sp(2n,\mathbb{C})$ with its two canonical real forms we were able to show an embedding of $J(V,\omega,*)$ in $Sp(n)/U(n)$, a flag manifold having all $J(V,\omega,l)$ as disjoint flag domains. We defer the proof to appendix A.

Conjugating by some non-real group element yields the corollary: all $J(V,\omega,l)$ are Stein spaces. Recall $J(V,\omega,0)$ is Stein because $\mathcal{D}_n$ is convex and being Stein is preserved by biholomorphism.

There is an easier way to accomplish our objectives. There is a commutative holomorphic diagram

$$\begin{array}{ccc} J(V,\omega,l) & \stackrel{\phi}{\longrightarrow} & Gr(n,V^c) \\ & \searrow \quad \nearrow & \\ & Sp(n)/U(n) & \end{array}$$

where the map on the top is $J \mapsto V'' = V^-$. We shall call "real lagrangians" the n-dimensional $\omega$-isotropic $\mathbb{C}$-subspaces $W$ of $V^c$ such that $W \cap \overline{W} = 0$.

**Proposition 2.4.** *The map $\phi$ is a holomorphic embedding and has image the locally closed manifold $\mathbb{R}Lag(n,V^c)$ of real lagrangian subspaces.*

*Proof.* Since $\overline{V''} = V'$, the map is injective, and by definition $V'' = \phi(J)$ is isotropic. Now let $W \in Gr(n,V^c)$. To any $g \in GL(V^c,\mathbb{C})$ we associate a sequence

$$W \xrightarrow{g_|} V^c \xrightarrow{p} \frac{V^c}{W}$$

with $p$ only depending on $W$. Therefore

$$Gr(n,V^c) = \frac{GL(V^c,\mathbb{C})}{\{\underline{g}=0\}}$$

where $\underline{g} = p \circ g_|$. Thus

$$T_W Gr = \frac{\mathfrak{gl}(V^c,\mathbb{C})}{\{\underline{X}=0\}} \simeq Hom\left(W, \frac{V^c}{W}\right)$$

where $\simeq$ stands for $\underline{X} \simeq p \circ X_{|W}$. Now for real $g \in GL(V,\mathbb{R})$

$$\phi(g \cdot J) = \left\{v: \ gJg^{-1}(v) = -iv\right\} = gV''.$$

Hence $\mathrm{d}\phi : \mathfrak{m}_J \to T_{V''}Gr$ satisfies $\mathrm{d}\phi(A) = \underline{A}$ and so

$$\mathrm{d}\phi(JA) = -\mathrm{d}\phi(AJ) = -p \circ AJ_{|V''} = i\,p \circ A_{|V''} = i\,\mathrm{d}\phi(A).$$

Notice we proved the whole embedding of $\mathcal{J}(V)$ in $Gr(n,V^c)$ is holomorphic.

Now assume $W \in \mathbb{R}Lag(n,V^c)$. Clearly $\omega : V^c \times V^c \to \mathbb{C}$ is non-degenerate, so the maximal dimension an isotropic subspace can attain is precisely $n$. Indeed, we have a general formula, $\dim W + \dim W^\omega = 2n$, where $W^\omega$ is the $\omega$-anihilator of $W$.

With the above one proves that the hemi-symmetric form on $W$ defined by

$$\begin{aligned} h(w_1, w_2) &= i\omega(w_1, \overline{w}_2) \\ &= -i\omega(\overline{w}_2, w_1) = \overline{h(w_2, w_1)} \end{aligned}$$

is non-degenerate for real lagrangian $W$. According to the signature of this pseudo-metric we may then define $J \in J(V, \omega, l)$ by $Jw = -iw$, $\forall w \in W$, and $J\overline{w} = i\overline{w}$, hence such that $\phi(J) = W$. It is trivial to see $J$ is real. For example

$$\overline{J}\overline{w} = \overline{Jw} = i\overline{w} = J\overline{w}.$$

We have proved $\phi$ is a biholomorphism onto the aforesaid manifold. Notice also

$$\mathbb{R}Lag(n, V^c) \;=\; \{W : \; W \cap \overline{W} = 0\} \cap \{W : W = W^\omega\}.$$

On the right hand side, the first set is open in the grassmannian and the second is closed. ■

The above is only part of either the cell or the algebraic structures of the grassmannian. We will not pursue these in this work. As an example, in $V^c = \mathbb{C}^2$ every line is lagrangian and there is a circle $S^1$ in $P^1(\mathbb{C})$ of non-real lines. The open hemispheres are the two Siegel domains $\mathcal{D} = J(\mathbb{R}^2, \omega, 0)$ and $-\mathcal{D} = J(\mathbb{R}^2, \omega, 1)$.

## 2.2 Twistor space theory

We have finally come to the point where substantial preparatory material has been examined thoroughly in order to introduce the theory of twistor spaces according to [20, 22]. The reader will find here references to materia from all previous sections.

Having presented the standard fibre $\mathcal{J}(V)$ of a general twistor space, plus an important subspace of it, we now go back to the point in [20] where the “bundle of complex structures and its almost-complex structure” is introduced. This is the theory which will guide us through the study of the twistor space of a symplectic manifold.

Let $M$ be a $\mathrm{C}^\infty$-manifold of even dimension. Consider the referred subbundle of $\mathrm{End}\,TM$, which is the manifold defined as

$$\mathcal{J}(M) = \bigcup_{x\in M} \mathcal{J}(T_xM),$$

and let $\pi : \mathcal{J}(M) \to M$ be the projection map. We have then an associated natural short exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow T\mathcal{J}(M) \xrightarrow{\mathrm{d}\pi} E \longrightarrow 0 \tag{2.2}$$

of vector bundles over $\mathcal{J}(M)$, where $E = \pi^*TM$ and $\mathcal{V} = \ker \mathrm{d}\pi$.

At a point $J \in \mathcal{J}(M)$ such that $\pi(J) = x$ the fibre of $E$ is $T_xM$, on which $J$ acts as a linear endomorphism. Thus we have a canonical section $\Phi$ of $\mathrm{End}\,E$ given by

$$\Phi_{_J} = J.$$

On the other hand, each fibre of $\pi$ is a complex manifold by the results of section 2.1, so we can identify the vertical tangent distribution $\mathcal{V}$ to the complex vector bundle whose fibre is

$$[J, \mathfrak{gl}(T_xM)] = \{A \in \mathfrak{gl}(T_xM) : \; AJ = -JA\}\,.$$

Since $\mathfrak{gl}(T_xM) = \mathrm{End}\,E_{_J}$, we may further regard $\mathcal{V}$ as the subbundle $[\Phi, \mathrm{End}\,E]$ of $\mathrm{End}\,E$ consisting of endomorphisms which anticommute with $\Phi$. Consequently, $\mathcal{V}$ has an endomorphism $\mathcal{J}^v$ with square $-1$ which coincides with the complex structure of each fibre. When we view $\mathcal{V}$ as the subbundle of $\mathrm{End}\,E$ thus described, $\mathcal{J}^v$ is left multiplication by $\Phi$.

We see that both $\mathcal{V}$ and $E$ admit complex structures. We shall now make use of a linear connection on $M$ to create a splitting of (2.2). This will then lead us to a ‘direct sum’ almost-complex structure on $\mathcal{J}(M)$.

Let $\nabla$ be a connection on $TM$. It induces a connection $\pi^*\nabla$ in $E = \pi^*TM$ and also in $\operatorname{End} E$. The following propositions 2.5, 2.6 and theorem 2.1 are taken from [20]. For the sake of understanding and completeness of our exposition we copy also their proofs.

**Proposition 2.5.** $\mathcal{H}^\nabla = \{X \in T\mathcal{J}(M) : \ (\pi^*\nabla)_X\Phi = 0\}$ *is a complement for* $\mathcal{V}$ *in* $T\mathcal{J}(M)$.

*Proof.* "Choose a vector space $V$ of the same dimension as $M$ and let $F(M)$ be the frame bundle of $M$, consisting of all linear isomorphisms

$$p : V \longrightarrow T_xM.$$

This is a principal $GL(V)$ bundle over $M$, and $\nabla$ is determined by the $\mathfrak{gl}(V)$-valued 1-form $\alpha$ on $F(M)$ given by

$$\nabla_X(sv) = s(s^*\alpha)(X)v$$

for any $v \in V$ and any section $s$ of $F(M)$, and

$$\alpha(\tilde{A}) = A$$

where $A \in \mathfrak{gl}(V)$ and $\tilde{A}$ is the vector field

$$\tilde{A}_p = \frac{\mathrm{d}}{\mathrm{d}t}_{|0}\, p \circ \exp tA.$$

$\ker\alpha$ is a horizontal distribution on $F(M)$. Fixing $J_0 \in \mathcal{J}(V)$, we get a map

$$\pi_1 : F(M) \longrightarrow \mathcal{J}(M) \tag{2.3}$$

given by

$$\pi_1(p) = pJ_0p^{-1}.$$

The derivative of $\pi_1$ maps the horizontal distribution on $F(M)$ onto a horizontal distribution on $\mathcal{J}(M)$. The proposition will be proved by showing that this distribution coincides with $\mathcal{H}^\nabla$.

Now (2.3) is a principal $GL(V, J_0)$ bundle over $\mathcal{J}(M)$ which is a reduction of $\pi^*F(M)$. Thus $E$ is associated to (2.3) with fibre $V$. Since $\Phi$ is a section of $\operatorname{End} E$ it corresponds with an equivariant function $\hat{\Phi}$ on $F(M)$ given by

$$\hat{\Phi}(p) = p^{-1}\Phi_{\pi_1(p)}p = J_0$$

and thus constant. But $(\pi^*\nabla)\Phi$ corresponds to the 1-form

$$\mathrm{d}\hat{\Phi} + [\alpha, \hat{\Phi}].$$

Since $\hat{\Phi}$ is constant this is just $[\alpha, \hat{\Phi}]$. If $Y$ is in $T_pF(M)$ and $X = \mathrm{d}\pi_1(Y)$ then $(\pi^*\nabla)_X\Phi$ corresponds with the endomorphism $[\alpha_p(Y), J_0]$. If $Y$ is horizontal this gives $(\pi^*\nabla)_X\Phi = 0$ so $X \in \mathcal{H}^\nabla$. If $Y$ is vertical for the projection onto $M$ then $[\alpha_p(Y), J_0] = 0$ if and only if $\alpha_p(Y) \in \mathfrak{gl}(V, J_0)$ and this in turn happens if and only if $Y \in \ker \mathrm{d}\pi_{1p}$. Since $\mathrm{d}\pi_1$ is surjective this gives the result." ([20, pages 37–38].) ■

Let

$$P : T\mathcal{J}(M) \longrightarrow \mathcal{V}$$

be the projection onto $\mathcal{V}$ with kernel $\mathcal{H}^\nabla$. We may view it both as a $\mathcal{V}$- and as a $\mathrm{End}\, E$-valued 1-form on $\mathcal{J}(M)$. Now, implied by the previous reasons, we find that $(\pi^*\nabla)\Phi$ corresponds with the 1-form $[\alpha, \hat{\Phi}]$ on $F(M)$. In terms of the bundle map $P$, this gives place to the next conclusion (note that we are just following [20]).

**Proposition 2.6.** $(\pi^*\nabla)\Phi = [P, \Phi] = 2P\,\Phi$.

Since $\mathcal{H}^\nabla$ is complementary to $\mathcal{V}$, $\mathrm{d}\pi : \mathcal{H}^\nabla \to E$ is an isomorphism so we can transport $\Phi$ from $E$ to $\mathcal{H}^\nabla$ to give an endomorphism $\mathcal{J}^h$ of $\mathcal{H}^\nabla$ with square $-1$ and

$$\mathcal{J}^\nabla = (\mathcal{J}^v, \mathcal{J}^h)$$

is then the referred almost complex structure on $\mathcal{J}(M)$ — which depends only on the choice of the connection on $TM$.

Note also that the above proof shows that

$$T\mathcal{J}(M) \;=\; F(M) \times_{GL(V,J_0)} \mathfrak{m}_{J_0} \oplus V.$$

So $\mathcal{J}^\nabla$ corresponds with the constant function on $F(M)$ which is left multiplication by $J_0$ on $\mathfrak{m}_{J_0}$ and $J_0$ itself on $V$. It follows that the $(0,1)$-tangents for $\mathcal{J}^\nabla$ are identified with the bundle associated to $F(M)$ with $\mathfrak{m}^-_{J_0} \oplus V^-$ as fibre.

As presented once in (1.9), consider now the $V$-valued 1-form $\theta$ on $F(M)$ defined by

$$\theta_p(X) = p^{-1}\mathrm{d}\pi_0(X),$$

where $\pi_0 : F(M) \to M$ is the projection, and put

$$\theta^{\pm} = J_0^{\pm}\theta.$$

Let

$$\alpha = \alpha^0 + \alpha^+ + \alpha^-$$

be the decomposition of the connection form $\alpha$ of $\nabla$ relative to the decomposition as in proposition 2.1 of $\mathfrak{gl}(V)^c$:

$$\mathfrak{gl}(V)^c = \mathfrak{gl}(V, J_0)^c + \mathfrak{m}^+_{J_0} + \mathfrak{m}^-_{J_0}.$$

Then the components of $\alpha^+, \theta^+$ span the pull-backs to $F(M)$ of (1,0) forms on $\mathcal{J}(M)$. This is a consequence of the (0,1) tangents of $\mathcal{J}(M)$ having $\mathfrak{m}^-_{J_0} \oplus V^-$ as fibre.

**Theorem 2.1.** *The almost-complex structure $\mathcal{J}^\nabla$ of $\mathcal{J}(M)$ is integrable if and only if the torsion $T$ and curvature $R$ of $\nabla$ satisfy*

$$J^+T(J^-X, J^-Y) = 0, \qquad J^+R(J^-X, J^-Y)J^- = 0,$$

*for all $X, Y \in TM$, $J \in \mathcal{J}(M)$.*

*Proof.* "We view the torsion and curvature of $\nabla$ as 2-forms on $F(M)$ with values in $V$ and $\mathfrak{gl}(V)$ respectively. They are given by [*cf.* proposition 1.7 and formula (1.6)]. It is a standard result that an almost-complex structure is integrable if and only if the (1,0) forms generate a d-closed ideal. This will be so if and only if their pull-backs to $F(M)$ also generate a d-closed ideal. By [the previous remarks], $\mathcal{J}^\nabla$ is integrable if and only if the components of $\alpha^+$ and $\theta^+$ generate a d-closed ideal. But, for instance,

$$\begin{aligned}\mathrm{d}\theta^+ \;=\; J_0^+\mathrm{d}\theta \;&=\; J_0^+\tau - J_0^+\alpha \wedge \theta \\ &=\; J_0^+\tau - \alpha^+ \wedge \theta - J_0^+\alpha \wedge \theta^+.\end{aligned}$$

Thus $\mathrm{d}\theta^+$ is in the ideal if and only if $J_0^+\tau$ is. Likewise

$$\mathrm{d}\alpha^+ = J_0^+\mathrm{d}\alpha J_0^- = J_0^+\rho J_0^- - J_0^+\alpha\wedge\alpha J_0^-$$

and

$$\begin{aligned} J_0^+\alpha\wedge\alpha J_0^- &= J_0^+\alpha\wedge J_0^+\alpha J_0^- + J_0^+\alpha J_0^-\wedge\alpha J_0^- \\ &= J_0^+\alpha\wedge\alpha^+ + \alpha^+\wedge\alpha J_0^- . \end{aligned}$$

Thus $\mathrm{d}\alpha^+$ is in the ideal if and only if $J_0^+\rho J_0^-$ is. Hence $\mathcal{J}^\nabla$ is integrable if and only if $J_0^+\tau$ and $J_0^+\rho J_0^-$ are in the ideal generated by $\theta^+$ (since only horizontal forms are involved). Hence the theorem." ([20, pages 39–40].) ■

It is the almost-complex manifold $(\mathcal{J}(M), \mathcal{J}^\nabla)$ which is called the twistor space of $M$ (in this thesis the adjective 'general' is used before, to stress that this is the space of *all* complex structures).

A conclusion we may draw is that the theory above is the most correct in which to consider and solve those problems raised, already, by twistor spaces. Since, between many other and by following [20] again, we find that the theory generalises naturally to several subspaces. This accounts for the length of our presentation.

The cited article continues now with a detailed analysis of the representation theory involved in conditions like those of theorem 2.1. In particular it was discovered that if $\mathcal{J}^\nabla$ is integrable then one may change $\nabla$ to a torsion free connection inducing the same $\mathcal{J}^\nabla$.

In [22, 23] a second almost-complex structure on $\mathcal{J}(M)$ was considered, defined by

$$\mathcal{J}_2^\nabla = (-\mathcal{J}^v, \mathcal{J}^h),$$

from which several results on the theory of harmonic maps were obtained, some based on the fact that this structure is never integrable. We will see later a particular situation where one can prove the latter in a very intuitive way (*cf.* proposition 3.1). But such claim, well known for the twistor space of a riemannian manifold, is much more essential.

**Proposition 2.7.** *If $\nabla$ is torsion free, $\mathcal{J}_2^\nabla$ is not integrable.*

*Proof.* Following the last proof, suppose $\alpha^-$ and $\theta^+$ generate the d-closed ideal of (1,0)-forms. Then

$$\begin{aligned} \mathrm{d}\theta^+ &= -J_0^+\alpha\wedge\theta \\ &\equiv -J_0^+\alpha J_0^-\wedge\theta \;\equiv\; 0 \bmod \{\alpha^-,\theta^+\} \end{aligned}$$

Taking any (0,1)-horizontal vector $u$ and any (0,1)-vertical $v$ in $TF(M)^c$, *i.e.* corresponding to some vectors in $FM\times_{GL(V,J_0)}\mathfrak{m}^+_{J_0}\oplus V^-$, then the above is equivalent to

$$J_0^+\alpha(v)J_0^-\theta(u)=0$$

or $J_0^+\alpha(v)J_0^-=0$, which is absurd because $\mathfrak{m}^+$ is not 0. ∎

In the following theorem we rewrite another one from [20] in a way it serves our purposes more directly. One can consider it a very, very simple corollary. Recall $\mathcal{H}^\nabla$ denotes ker $\pi^*\nabla.\,\Phi$.

**Theorem 2.2.** *Let $Z$ be an almost-complex manifold and*

$$\pi: Z\longrightarrow M$$

*be a smooth submersion onto $M$ with fibres which are smoothly varying complex manifolds. Suppose that $Z$ has a horizontal distribution $\mathcal{H}^Z$ which is $j$-related to the horizontal distribution $\mathcal{H}^\nabla$ of a connection $\nabla$ on $TM$ via a pseudo-holomorphic smooth fibre preserving map*

$$j: Z\longrightarrow \mathcal{J}(M).$$

*Then integrability of $J^Z$ implies that the torsion $T$ and curvature $R$ of $\nabla$ satisfy*

$$J^+T_x(J^-X,J^-Y)=0,\qquad\qquad J^+R_x(J^-X,J^-Y)J^-=0$$

*for all $J\in j(Z)$ and $X,Y\in T_xM$. If $j$ is an immersion these conditions are also sufficient.*

This result can be applied in the following situation. Suppose $M$ has a $G$-structure in the sense that $TM$ is associated to some principal $G$-bundle $P$, and some representation $V$ of $G$. If $Y$ is a complex homogeneous $G$-manifold and

$$j_0 : Y \longrightarrow \mathcal{J}(V)$$

is a holomorphic and $G$-equivariant map, then we get a map

$$\begin{array}{ccc} P \times_G Y & \xrightarrow{j} & \mathcal{J}(M) \\ & & \downarrow \pi \\ & & M. \end{array}$$

Any connection on $P$ induces horizontal distributions on $Z = P \times_G Y$ and $\mathcal{J}(M) = F(P/G) \times_G \mathcal{J}(V)$ which are preserved by $j$, and the theorem above gives the condition for the integrability of the associated almost-complex structure $J^Z$. This *associated* almost-complex structure is constructed exactly as in the twistor space by means of the map $j_0$, thus making $j$ pseudo-holomorphic.

In an eclectic style, one could try to sort the relations between theorem 1.2 and theorem 2.2, but one has to beware that $G$ here is not necessarily complex.

After the above construction, the theory in [20] follows through the consideration of several particular cases: the riemannian twistor space, the almost-hermitian, the quaternionic and even the octonionic structures. It omits the symplectic case, which we wish to study. Before we go into it let us see a result which will become obvious in section 3.1.

**Corollary 2.1.** *If $\mathcal{J}^\nabla$ is integrable and $\sigma \in Diff(M)$ is a diffeomorphism preserving the $G$-structure, then $\mathcal{J}^{\sigma\cdot\nabla}$ is integrable.*

*Proof.* Assuming that $\sigma$ admits some 'gauge lift' along $\sigma$ to $P$ commuting with the projection map, then we will have $\sigma \cdot J$ a complex structure and $\sigma \cdot J^\pm = (\sigma \cdot J)^\pm$ for any section $J$ in the image of $j$. The result follows hence from the identities $T^{\sigma\cdot\nabla} = \sigma \cdot T$ and $R^{\sigma\cdot\nabla} = \sigma \cdot R$. ∎

Following the original idea of I. Vaisman (*cf.* [26]) we shall now apply all the above to the case where $(M,\omega)$ is a symplectic $2n$-real manifold, $G = Sp(V,\omega)$, $F^sM$ is the symplectic frame bundle, $Y = J(V,\omega,l)$ with its $G$-complex structure descending from that of $\mathcal{J}(V)$ and $\nabla$ is a symplectic connection (torsion free).

Hence, with $\mathcal{J}(M,\omega,*) = Z$ as above, $j_0$ is just the inclusion map, and then as in theorem 2.2 we have

$$\begin{array}{ccc} \mathcal{J}(M,\omega,l) & \xrightarrow{j} & \mathcal{J}(M) \\ \pi\searrow & & \swarrow \\ & M. & \end{array}$$

Of course we have identities

$$\mathcal{J}(M,\omega,l) = \Big\{ J \in \mathcal{J}(M) :\ \omega = \omega^{1,1} \text{ and } g_{_J} \text{ has sign. } (2n-2l,2l) \Big\}$$

and

$$\pi^{-1}(x) = J(T_xM,\omega_x,l).$$

Henceforth the integrability condition for $\mathcal{J}^\nabla$ is given by theorem 2.2, since $j$ is the inclusion map. The condition has been further dismantled in the following result due to F. Burstall and J. Rawnsley, after a mistaken attempt of I. Vaisman (*cf.* [26]).

**Theorem 2.3.** *$\mathcal{J}^\nabla$ is integrable iff $\nabla$ is of Ricci type.*

For the proof see references in [8]. As we state it, the result implies that if one of the $n+1$ $\mathcal{J}(M,\omega,l)$ is complex then they all are. Since this known property seems not to have been written down elsewhere we give our proof in appendix A.

Starting with a complex structure on the *twistor space of a symplectic manifold* it would be very interesting to be able to recover a symplectic connection on $M$

inducing the same twistor structure. Such connection would thus satisfy the field equations (1.14).

The theorem looks very much like its riemannian counterpart (see [3, 20]) since, as the reader may recall, Ricci type is the symplectic analogue of a self dual riemannian 4-manifold or conformally flat in dimensions $>4$. Something particular to the symplectic case is given next.

**Theorem 2.4.** *If* $\mathcal{J}^{\nabla^1}=\mathcal{J}^{\nabla^2}$ *then* $\nabla^1=\nabla^2$.

*Proof.* Let $A=\nabla^2-\nabla^1$ and define $\underline{A}\in\Gamma S^3T^*M$ by

$$\begin{aligned}\underline{A}(X,Y,Z) &= \omega(A(X)Y,Z)\\ &= \omega(A(Y)X,Z) \;=\; \omega(A(Z)Y,X)\end{aligned}$$

for all $X,Y,Z\in TM$.

Now let $X\in\mathcal{H}^{\nabla^1}$. Then $X=X_2+Y$, with $X_2\in\mathcal{H}^{\nabla^2}$, $Y\in\mathcal{V}$. By propositions 2.5 and 2.6 we have

$$[Y,\Phi]=\pi^*\nabla^2_X\Phi=[\pi^*A(\mathrm{d}\pi X),\Phi].$$

Suppose further that $\mathcal{J}^{\nabla^1}=\mathcal{J}^{\nabla^2}$ and $u$ is a (1,0)-$\nabla^1$-horizontal vector field. Then $u=u_2+v$, with $v$ a (1,0)-vertical vector field because the almost-complex structure on the fibre of the twistor space is always the same. Hence

$$[v,\Phi]=[\pi^*A(\mathrm{d}\pi\, u),\Phi]\ \text{ is }\ (1,0),\qquad \forall u\in\mathcal{H}^{\nabla^1\,(1,0)},$$

since we have noticed before that $[\mathfrak{gl}(V,J)^c,\mathfrak{m}^+_J]\subset\mathfrak{m}^+_J$. In the base manifold this translates as

$$[A_x(J^+X),J]\ \text{ is }\ (1,0),\qquad \forall J\in\pi^{-1}(x),\ \forall X\in T_xM.$$

Equivalently this means the projection to every $\mathfrak{n}^c_J$ is (1,0), or

$$J^-A(J^+X)J^+=0$$

(*cf.* proposition 2.1). From here to the equality $\underline{A}^{3,0}=0$, $\forall J$, is immediate. This says $\underline{A}$ must take values in the largest $G$-invariant subspace of symmetric tensors which satisfy

$$\underline{A}(J_0^+X,J_0^+Y,\ldots)=0$$

for some fixed $J_0$ and all $X, Y, \ldots$. Indeed, since any $J = gJ_0g^{-1}$ for some $g \in G$, we will also have $J^+ = \frac{1}{2}(1 - iJ) = gJ_0^+g^{-1}$ and therefore

$$\begin{aligned} 0 = (g^{-1} \cdot \underline{A})(J_0^+X, J_0^+Y, \ldots) &= \underline{A}(gJ_0^+X, gJ_0^+Y, \ldots) \\ &= \underline{A}(J^+gX, J^+gY, \ldots) \end{aligned}$$

But $S^k(V^+) = S^k(\mathbb{R}^{2n})$ is irreducible under $G$ for all $k$, so $\underline{A}$ must be 0. ■

We would still like to remark that proposition 2.7 and theorem 2.4, for which there are no references, are due to J. Rawnsley.

## 2.3 Examples

In the general programme of [20] a twistor space over some base space $M$ is an almost-complex manifold $Z$ together with a submersion

$$f : Z \longrightarrow M$$

with fibres almost-complex submanifolds. For each $z$ in the fibre $Z_x = f^{-1}(x)$ we have an isomorphism

$$\frac{T_zZ}{\mathcal{V}_z} \longrightarrow T_xM$$

where $\mathcal{V}_z = \ker \mathrm{d}f_z = T_zZ_x$. Since the vector space $T_zZ/\mathcal{V}_z$ is complex, we can bring this complex structure to $T_xM$ in order to construct a map

$$j : Z \longrightarrow \mathcal{J}(M).$$

Of course $f$ is a pseudo-holomorphic map with respect to some structure on $M$ if, and only if, $j$ is constant along the fibres.

If $(M,\omega)$ is a symplectic manifold we shall call $Z$ an "$\omega$-twistor space" if the image of $j$ is in $\mathcal{J}(M,\omega,*)$. For example, given a symplectic connection $\nabla$ on $M$, the tautology of the definition of $\mathcal{J}^\nabla$, essentially

$$\mathcal{J}^\nabla_{|\mathcal{H}_J} = J,$$

proves $\mathcal{J}(M,\omega,*)$ to be a true $\omega$-twistor space over $M$.

We have seen that the Siegel domain is non-compact and that it embeds holomorphically in a *complex* manifold, namely the grassmannian (*cf.* proposition 2.4). Thus we may ask if it is possible to embed $\mathcal{J}(M,\omega,*)$ holomorphically into a bigger space; for example, into a 1-point compactification, and, in particular, into other bundles over $M$. Well, $\nabla$ is a linear connection so the whole $\mathcal{J}(M)$ extends $\mathcal{J}(M,\omega,*)$. But if we demand compactness what should happen?

For example, we ask for an extension of $\mathcal{J}^\nabla$ to the compact $Sp(n)/U(n)$-bundle of $\mathbb{C}$-lagrangian $n$-planes over the real symplectic $2n$-manifold $M$. Unfortunately, such extension does not exist.

**Proposition 2.8.** *It is not possible to extend $(\mathcal{J}(M,\omega,*),\mathcal{J}^\nabla)$ to a bigger almost-complex manifold, of the same dimension, which is also a fibre bundle over $M$.*

*Proof.* By fibre bundle we mean a submersion (*cf.* [14]). Assuming the extension to a space $Z$ exists, the theory above yields a continuous map

$$j : \overline{\mathcal{J}(M,\omega,*)} \longrightarrow \mathcal{J}(M)$$

on the closure of $\mathcal{J}(M,\omega,*)$ in $Z$, because, letting $z$ be any point on the boundary of the $\omega$-twistor space, projecting to a point $x \in M$, then $T_zZ_x$ is still a complex vector space.

Also by continuity we have that $\omega = \omega^{1,1}$ for $j(z)$. But since $j$ is the identity in $\mathcal{J}(M,\omega,*)$ we arrive at a contradiction. ∎

Regarding a matter of different nature, it seems to us that the 'non-constant' compact $\omega$-twistor spaces are not easy to construct or describe.

**Proposition 2.9.** *There are no $\omega$-twistor spaces with compact fibres of* $\dim > 0$ *satisfying the hypothesis of theorem 2.2 and with the map $j$ an immersion.*

*Proof.* Assuming $Z$ were such a space, then

$$j : Z \longrightarrow \mathcal{J}(M, \omega, *)$$

would be holomorphic when restricted to each fibre. But the Siegel domain is a Stein manifold so its compact analytic submanifolds are points. ■

Clearly the proposition avoids the holomorphic case, which induces a map $j$ *constant* along the fibres.

**Remark.** Here is an attempt to produce 'non-constant' $\omega$-twistor spaces. Consider a compact symplectic fibration $f : Z \to M$ (*cf.* [18]). There is a theorem of W. Thurston stating that, under certain hypothesis of the topological kind, we can build a symplectic structure on $Z$ agreeing with the symplectic structures on the fibres of $Z$ (*cf.* [18]). Recall also that every symplectic manifold is almost-complex for many compatible complex structures, so we can choose one $J^Z$ on $Z$ such that all the fibres become almost-complex submanifolds. Because the fibres are symplectic and thus

$$T_z Z / \mathcal{V}_z$$

is symplectic 'for' $\pi^*\omega$, we find that the induced map $j$ will be $\mathcal{J}(M, \omega, *)$-valued, *i.e.* that $Z$ is an $\omega$-twistor space, willingly non-constant.

For example, consider $Z = P^n(\mathbb{C}) \times P^p(\mathbb{C}) \xrightarrow{f} P^n(\mathbb{C})$. Then it may be possible that $Z$ admits a compatible almost-complex structures for which the $\{z\} \times P^p(\mathbb{C})$ are almost-complex but the projection $f$ is not holomorphic.

Reassuringly, our attempt is not meaningless. In the end of this section we will be able to show an example.

The promised examples of twistor spaces of a symplectic manifold are about to be presented.

**Example 1.** Let $M = \mathbb{R}^2$, $\omega$ the canonical symplectic form, $\nabla$ any symplectic connection on $M$ — see proposition 1.12, from which we use the descriptions and notations in what follows. We want to describe $\mathcal{Z} = \mathcal{J}(M, \omega, 0)$ in terms of holomorphic charts, since $\mathcal{J}^\nabla$ is always integrable. There is a simple way to see this: $R^\nabla$ is a 2-form, so it is proportional to $\omega$. Since $\omega = \omega^{1,1}$ for $J \in \mathcal{Z}$, we have $R^\nabla(J^-\ ,J^-\ ) = 0$, and then we apply theorem 2.1 to prove the claim. Otherwise one can rely on theorem 2.3 and recall from section 1.4 that the Weyl part of the curvature is always zero in the two dimensional case.

Now suppose $v \in T^{0,1}M = T'' = T^-$ for $J$. If $v = \frac{\partial}{\partial z}$ then $J \in -\mathcal{Z}$, so we may already assume, up to a scalar,

$$v = \frac{\partial}{\partial \overline{z}} + w\frac{\partial}{\partial z}$$

for some $w \in \mathbb{C}$. The positive condition reads

$$-i\omega(v, \overline{v}) < 0.$$

Since

$$\begin{aligned} -i\omega(v,\overline{v}) &= \frac{1}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z}\left(\frac{\partial}{\partial \overline{z}} + w\frac{\partial}{\partial z}, \frac{\partial}{\partial z} + \overline{w}\frac{\partial}{\partial \overline{z}}\right) \\ &= \frac{1}{2}(w\overline{w} - 1), \end{aligned} \tag{2.4}$$

we recover the Siegel domain $\mathcal{D} = \{w :\ |w| < 1\}$. Because $TM$ is $\mathrm{C}^\infty$-trivial we have

$$\mathcal{Z} = M \times \mathcal{D} \stackrel{\pi}{\longrightarrow} M.$$

Now working together with $T\mathcal{Z}^c$ let

$$u = \frac{\partial}{\partial \overline{z}} + w\frac{\partial}{\partial z} + b\frac{\partial}{\partial w} + c\frac{\partial}{\partial \overline{w}}$$

be a $\mathcal{J}^\nabla$-(0,1)-horizontal vector field, thus projecting to $v=\mathrm{d}\pi(u)$ and where $w$ is the fibre variable. Recall the canonical section $\Phi\in\Gamma(\mathrm{End}\,\pi^*TM)$ defined by $\Phi_{_J}=J$. Then

$$\Phi v=-iv$$

where we see $v$ as a (0,1)-section of $(\pi^*TM)^c$. We can compute the function $b$ solving

$$(\pi^*\nabla_u\Phi)\,v=0. \tag{2.5}$$

On the left hand side we have — recall proposition 1.12 —

$$\begin{aligned}
(\pi^*\nabla_u\Phi)\,v &= \pi^*\nabla_u\Phi v-\Phi\,\pi^*\nabla_u v\\
&= -(i+\Phi)\pi^*\nabla_u v\\
&= -(i+\Phi)\left(\nabla_{\mathrm{d}\pi(u)}\frac{\partial}{\partial\overline{z}}+u(w)\frac{\partial}{\partial z}+w\nabla_{\mathrm{d}\pi(u)}\frac{\partial}{\partial z}\right)\\
&= -(i+\Phi)\left(\nabla_{\partial_{\overline{z}}}\partial_{\overline{z}}+w\nabla_{\partial_z}\partial_{\overline{z}}+b\frac{\partial}{\partial z}+w\nabla_{\partial_{\overline{z}}}\partial_z+w^2\nabla_{\partial_z}\partial_z\right)\\
&= -(i+\Phi)\left(\overline{\alpha}\frac{\partial}{\partial\overline{z}}+\overline{\beta}\frac{\partial}{\partial z}-\overline{\alpha}w\frac{\partial}{\partial z}-\alpha w\frac{\partial}{\partial\overline{z}}\right.\\
&\qquad\qquad\left.+b\frac{\partial}{\partial z}-\overline{\alpha}w\frac{\partial}{\partial z}-\alpha w\frac{\partial}{\partial\overline{z}}+w^2\alpha\frac{\partial}{\partial z}+w^2\beta\frac{\partial}{\partial\overline{z}}\right)\\
&= -(i+\Phi)\left((\overline{\beta}-2\overline{\alpha}w+b+w^2\alpha)\frac{\partial}{\partial z}+(\overline{\alpha}-2\alpha w+w^2\beta)\frac{\partial}{\partial\overline{z}}\right).
\end{aligned}$$

Hence (2.5) says we are in the presence of a (0,1)-vector for $J$, therefore, playing in dimension 1 as we are, there exists $\lambda\in\mathbb{C}$ such that

$$(\overline{\beta}-2\overline{\alpha}w+b+w^2\alpha)\frac{\partial}{\partial z}+(\overline{\alpha}-2\alpha w+w^2\beta)\frac{\partial}{\partial\overline{z}}=\lambda\left(\frac{\partial}{\partial\overline{z}}+w\frac{\partial}{\partial z}\right).$$

Henceforth

$$\overline{\beta}-2\overline{\alpha}w+b+w^2\alpha=\overline{\alpha}w-2\alpha w^2+w^3\beta$$

or

$$b=-\overline{\beta}+3\overline{\alpha}w-3\alpha w^2+\beta w^3.$$

After all, to find the function $c$ one would have to proceed as above but with (1,0)-vector fields. However, this will not be happening.

**Proposition 2.10.** *$f \in \mathcal{O}_{\mathcal{Z}}$ if and only if*

$$\begin{cases} \frac{\partial f}{\partial \overline{w}} = 0 \\ \\ \frac{\partial f}{\partial \overline{z}} + w\frac{\partial f}{\partial z} + \left(-\overline{\beta} + 3\overline{\alpha}w - 3\alpha w^2 + \beta w^3\right)\frac{\partial f}{\partial w} = 0. \end{cases}$$

*Proof.* Notice $\partial/\partial\overline{w}$ is a (0,1)-vector field tangent to the fibres of $\mathcal{Z}$, hence the first equation. The second is $u(f) = 0$. ■

By Darboux's theorem the proposition describes locally the twistor space of any Riemann surface.

A moments' thought on the results of proposition 2.3 and proposition 2.4 shows that the complex structure assumed above on the Siegel disk, the usual one, is the one which sets $\mathcal{J}^{\nabla}$ in accordance with the theory. We give an independent proof of integrability:

$$\left[\frac{\partial}{\partial\overline{w}}, u\right] = \frac{\partial b}{\partial\overline{w}}\frac{\partial}{\partial w} + \frac{\partial c}{\partial\overline{w}}\frac{\partial}{\partial\overline{w}} = \frac{\partial c}{\partial\overline{w}}\frac{\partial}{\partial\overline{w}}$$

is again a (0,1)-vector field. For the second almost-complex structure $\mathcal{J}_2^{\nabla}$ we have

$$\left[\frac{\partial}{\partial w}, u\right] = \frac{\partial}{\partial z} + \frac{\partial b}{\partial w}\frac{\partial}{\partial w} + \frac{\partial c}{\partial w}\frac{\partial}{\partial\overline{w}},$$

clearly not a linear combination of $\frac{\partial}{\partial w}$ and $u$. This proves proposition 2.7 in real dimension 2 — also for $\mathcal{J}(M, \omega, 1)$!

We have not explored the meaning of the nice cubic polynomial in $w$ appearing in the structure equations of the twistor space. It seems the latter could have an algebraic structure if $\nabla$ were an *algebraic* connection. On the other hand, we could not solve the equations even in some simple cases.

Notice that the complex structure of the fibres of $\mathcal{Z}$ obviously extends outside each one of them. Hence the pertinency of proposition 2.8.

**Example 2.** This is the trivial case: recall $\nabla = \mathrm{d}$ is symplectic because $M = \mathbb{R}^2$ is Kähler, so assume $\alpha = \beta = 0$. We have the following global chart for $\mathcal{Z}$:

$$\begin{aligned} \phi: \quad M \times \mathcal{D} &\longrightarrow \mathbb{C} \times \mathcal{D} \\ (z, w) &\longmapsto (w\overline{z} - z, w). \end{aligned}$$

This map is injective if and only if $|w| \neq 1$. Adding a point at infinity on the right hand side and recalling the grassmannian model of the twistor space, the same map composed with $1/w$ gives a chart of $\mathcal{J}(\mathbb{R}^2, \omega, 1)$. Curiously, this example is the only one for which the natural fibre chart $w$ is a *globally* holomorphic function.

Since $\mathbb{C} \times \mathcal{D}$ is convex, the connected components of $\mathcal{J}(\mathbb{R}^2, \omega, *)$ with complex structure arising from the trivial connection are Stein 2-manifolds. We shall look upon these interesting aspects in the next chapter.

Finally the theory says, and we have confirmed it elsewhere, that

$$\frac{\mathbb{C}^2}{\operatorname{span}\left(\frac{\overline{\xi}+\overline{w}\xi}{|w|^2-1}, 1\right)} \simeq \left(\mathbb{R}^2, \text{complex structure "}J = w\text{"}\right)$$

where $\phi(z, w) = (\xi, w)$.

The problem of finding charts for the flat torus or cylinder is still open. It seems to lead into deep Analysis.

**Example 3.** Consider $M = P^1(\mathbb{C}) = \mathbb{R}^2 \cup \{\infty\}$ with its Kähler metric and corresponding Levi-Civita connection, which is thus symplectic. The 2-form is

$$\omega = \frac{i}{2} \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 + |z|^2)^2}$$

so, proceeding as in (2.4), we describe the twistors' fibres over the open set $\mathbb{R}^2$ just as before. Following the theory presented in section 1.3 about hermitian manifolds the connection is type (1,0), *i.e.* transforms holomorphic sections in (1,0)-forms. Thus $\nabla$ on $T^*M$ is determined by

$$\nabla \mathrm{d}z = \alpha \, \mathrm{d}z \otimes \mathrm{d}z$$

and a conjugate version of this equation, bearing in mind $\nabla$ is real. Solving $\nabla\omega = 0$ leads to

$$\alpha = \frac{2\overline{z}}{1 + |z|^2}.$$

Proceeding then exactly as in example 1 we find: $f \in \mathcal{O}_{\mathcal{J}(M - \{\infty\}, \omega, 0)}$ if and only if

$$\begin{cases} \frac{\partial f}{\partial \overline{w}} = 0 \\ \\ \frac{\partial f}{\partial \overline{z}} + w \frac{\partial f}{\partial z} + \frac{2w(w\overline{z} - z)}{1+|z|^2} \frac{\partial f}{\partial w} = 0. \end{cases} \tag{2.6}$$

Let $(z_1, w_1)$ denote coordinates for the twistor space of $M$ minus the other pole. The affine transformation on the base $z_1 = \sigma(z) = 1/z$ can also be raised to a $\mathcal{J}^\nabla$-holomorphic transformation of the twistor space. $w_1$ is defined by requiring that

$$(\mathrm{d}\sigma)^c\left(\frac{\partial}{\partial\overline{z}} + w\frac{\partial}{\partial z}\right) = \lambda\left(\frac{\partial}{\partial\overline{z}_1} + w_1\frac{\partial}{\partial z_1}\right)$$

for some $\lambda \in \mathbb{C}$. That is, the real map $\mathrm{d}\sigma$ applies a (0,1)-$w$-vector into a (0,1)-$w_1$-vector. To solve the equation we only recall the useful formulas, valid in general,

$$\mathrm{d}\sigma\left(\frac{\partial}{\partial z}\right) = \frac{\partial\sigma}{\partial z}\frac{\partial}{\partial z_1} + \frac{\partial\overline{\sigma}}{\partial z}\frac{\partial}{\partial\overline{z}_1} = \overline{\mathrm{d}\sigma\left(\frac{\partial}{\partial\overline{z}}\right)}$$

which one may easily prove. Henceforth

$$w_1 = \frac{\overline{z}^2}{z^2}w$$

and we said $(z, w) \mapsto (z_1, w_1)$ is holomorphic because one verifies by straightforward computations that if a function $f(z_1, w_1)$ satisfies the system (2.6) in variables $(z_1, w_1)$ then

$$f\left(\frac{1}{z}, \frac{\overline{z}^2}{z^2}w\right)$$

also satisfies the linear system in variables $(z, w)$.

We shall see in section 3.1 that this last result is a consequence of $\nabla$ being $\sigma$-invariant. The latter can either be seen directly or deduced by uniqueness of the Levi-Civita connection after verifying $\sigma$ is an isometry. But this is immediate, since $\mathrm{d}z_1 = -\frac{1}{z^2}\mathrm{d}z$ and thus $\sigma \cdot \omega = \omega$.

**Example 4.** This is the generalisation of example 2. Let $M = \mathbb{R}^{2n}$ and

$$\omega = \frac{i}{2}\sum_k \mathrm{d}z_k \wedge \mathrm{d}\overline{z}_k.$$

We give a description of $\mathcal{J}(M, \omega)$ with complex structure arising from the trivial connection.

First notice that for any element $J$ we can find a basis of $T''M$ with vectors of the kind

$$v_k = \frac{\partial}{\partial\overline{z}}^k + \sum_l w_{kl}\frac{\partial}{\partial z}^l$$

with $k=1,\dots,n,\ w_{kl}\in\mathbb{C}$. For, if a linear combination of the $\partial/\partial z^l$ only were in $T''M$, then the positive condition would not be satisfied. Now, $\omega$ being (1,1) for $J$ implies

$$0=\omega(v_{k_1},v_{k_2})=-w_{k_1k_2}+w_{k_2k_1}.$$

The positive condition is given by

$$\begin{aligned} 0\ >\ -i\omega(v_k,\overline{v}_k) &= \frac{1}{2}\sum_l \mathrm{d}z_l\wedge\mathrm{d}\overline{z}_l\left(\frac{\partial}{\partial\overline{z}}^k+w_{kp}\frac{\partial}{\partial z}^p,\frac{\partial}{\partial z}^k+\overline{w}_{kq}\frac{\partial}{\partial\overline{z}}^q\right)\\ &= \frac{1}{2}\sum_l(-\delta_{kl}+w_{kl}\overline{w}_{kl})\\ &= \frac{1}{2}\left(-1+\sum_l|w_{kl}|^2\right)\end{aligned}$$

where repeated indices in $p,q$ have denoted a sum. With respect to the symmetric matrix $W=[w_{kl}]$ this is equivalent to $1-WW^*>0$ and so we meet another well known description of the Siegel domain $J(\mathbb{R}^{2n},\omega,0)$.

Continuing to reason as in example 1 we find that a function $f$ on the twistor space is holomorphic if $v_k(f)=0,\ \partial f/\partial\overline{w}^{pq}=0$. So a global chart for $\mathcal{J}(M,\omega)$ is given by the functions

$$f_{pq}=w_{pq}\qquad\text{and}\qquad f_k=\overline{z}_kw_{kk}-z_k$$

where $p\leq q$ and $1\leq k\leq n$.

**Example** of a compact $\omega$-twistor space: combining the previous example 1 with the method explained, that which uses a theorem of Thurston, we can now describe a $\omega$-twistor space which is a compact symplectic fibration.

Let $\mathbb{T}^2$ be the flat torus $\mathbb{R}^2/\mathbb{Z}^2$. Consider the real manifold

$$Z=\mathbb{T}^2\times P^1(\mathbb{C})\xrightarrow{\mathrm{pr}_1}\mathbb{T}^2$$

fibering over $\mathbb{T}^2$, with almost-complex structure $J^Z$ given by the following basis of (0,1)-tangents: the vectors

$$\frac{\partial}{\partial\overline{z}}+\frac{|t|}{1+|t|^2}\frac{\partial}{\partial z}\quad\text{and}\quad\frac{\partial}{\partial\overline{t}}.$$

$z$ is the usual chart of $\mathbb{R}^2$ and $t$ is a fixed affine coordinate of $P^1(\mathbb{C})$. Note that, for $t \neq 0$, we have

$$\frac{|\frac{1}{t}|}{1+|\frac{1}{t}|^2} = \frac{|t|}{1+|t|^2},$$

so $J^Z$ is well defined and preserves the natural splitting of $TZ$. Moreover, it is compatible with the canonical symplectic structure of $Z$. An easy computation shows that $J^Z$ is not integrable, but that is not important to our purposes.

Hence $Z$ is a twistor space, and, in fact, an $\omega$-twistor space. This is not much more difficult to see. The map $j : Z \to \mathcal{J}(\mathbb{T}^2,\omega) = \mathbb{T}^2 \times \mathcal{D}$ induced by $\mathrm{dpr}_1$ and the $\mathbb{C}$-vector bundle $TZ/\ker \mathrm{dpr}_1$ identifies with

$$j(z,t) = \left(z, \frac{|t|}{1+|t|^2}\right).$$

For the reader to compare with proposition 2.9, note that $j$ is not even open along the fibers $\{z\} \times P^1(\mathbb{C})$.

## 2.4 Maps into $\mathcal{J}(M,\omega,*)$

No matter which complex space, one always wants to understand either how it parametrises other spaces and how its subspaces look like. In the case of a twistor space, if the first problem seems to be difficult and dependent on the other space, the second appears more tractable in terms of the peculiar decomposition of the almost-complex structure $\mathcal{J}^\nabla$. Besides, this was already treated in [22] at least in the case of the riemannian twistor space. We say 'at least' because the results we are about to show, the study of maps into $\mathcal{J}(M,\omega,*)$, happened to be completely analogous to the results therein.

Consider a symplectic manifold $(M,\omega)$ and another space $N$, together with a smooth map $\psi$ inducing a diagram

$$\begin{array}{ccc} N & \xrightarrow{\psi} & \mathcal{J}(M,\omega,*) \\ & f \searrow & \downarrow \pi \\ & & M \end{array}$$

where $f$ is just $\pi\circ\psi$. Since $\psi(x)\in\pi^{-1}(f(x))$ is an $\omega$-compatible complex structure on each $T_{f(x)}M$ and since we have a natural vector bundles' transformation

$$\begin{array}{ccc} \psi^*E & \longrightarrow & TM \\ \downarrow & & \downarrow \\ N & \longrightarrow & M \end{array}$$

along $f$, where $E=\pi^*TM$ just like in section 2.2, we see $\psi$ as a $f^{-1}\omega$-compatible complex structure on $\psi^*E$, which we denote by $\Phi^\psi$. Hence

$$\Phi^\psi(x)=\Phi_{\psi(x)}=\psi(x).$$

Alternatively, we think of $\psi$ as the lagrangian $\mathbb{C}$-subbundle $\psi^+\subset\psi^*E^c$ over $N$. Denoting $E_j^+=\ker(j-i1)\subset E_j^c$ for any $j\in\mathcal{J}(M,\omega,*)$, we thus have

$$\psi^+=\psi^*E^+=\ker(\Phi^\psi-i1).$$

We now proceed to analyze the horizontal and vertical parts of $\mathrm{d}\psi$ induced by the connection-decomposition of the tangent bundle to the twistor space. Given a symplectic connection $\nabla$ on $TM$, the theory has shown us how to define a projection

$$P:T\mathcal{J}(M,\omega,*)\longrightarrow\mathcal{V}$$

with kernel $\mathcal{H}^\nabla$ and thus an isomorphism $T\mathcal{J}(M,\omega,*)\simeq\mathcal{V}\oplus E$. We have

$$\mathrm{d}\psi:TN\longrightarrow\psi^*T\mathcal{J}(M,\omega,*)=\psi^*\mathcal{V}\oplus\psi^*E,$$

and since $\mathrm{d}\pi\circ\mathrm{d}\psi=\mathrm{d}f$ the part in $\psi^*E$ is just $\mathrm{d}f$, whilst the part in $\psi^*\mathcal{V}$ can be identified with $\psi^*P$, viewing $P$ as a $\mathcal{V}$-valued 1-form on the twistor space. Thus the following result from [22] still holds.

**Proposition 2.11.** $\mathrm{d}\psi=(\psi^*P,\mathrm{d}f)$.

Since $[P,\Phi]=(\pi^*\nabla)\Phi$ and since the pull-back preserves all these relations in order to let us find that $[\psi^*P,\psi^*\Phi]=(\psi^*\pi^*\nabla)\psi^*\Phi$, we thus have always

$$[\psi^*P(X),\Phi^\psi]=(f^*\nabla)_X\Phi^\psi,\qquad \forall X\in TN.$$

Now, because the bracket with $\Phi$ is injective in $\mathcal{V}\subset \operatorname{End}E$, it follows $\mathrm{d}\psi$ is horizontal, *i.e.* $\psi^*P=0$, if and only if

$$(f^*\nabla)\Phi^\psi=0.$$

One might also want to recall that an equivalent characterisation of $\Phi^\psi$ being a parallel complex structure is given by the condition $f^*\nabla_X\Gamma\psi^+\subset\Gamma\psi^+,\ \forall X\in TN$.

**Remark.** To see a trivial application of the above consider a smooth section $j:M\rightarrow\mathcal{J}(M,\omega,*)$. We recall that $j$ may be integrable as a complex structure on $M$ but does not have to be parallel for $\nabla$. On the other hand, since $f=\pi\circ j=\mathrm{Id}$, $j$ is parallel iff it is horizontal, so the horizontal sections are precisely those for which the induced Levi-Civita connections of the metrics $g_j=\omega(\ ,j\ )$ are all the same, namely the given $\nabla$. We conclude that the non-kählerian symplectic manifolds cannot admit one globally horizontal section $j$ in spite of their many symplectic connections.

Notice furthermore that a horizontal section is always a pseudo-holomorphic map $(M,j)\xrightarrow{j}(\mathcal{J}(M,\omega,*),\mathcal{J}^\nabla)$. The proof is that

$$\begin{aligned}\mathcal{J}^h\mathrm{d}j_x &= \mathrm{d}\pi^{-1}j(x)\mathrm{d}\pi\,\mathrm{d}j_x\\ &= \mathrm{d}\pi^{-1}\mathrm{d}\pi\mathrm{d}j_x\,j(x)\\ &= \mathrm{d}j_x\,j(x).\end{aligned}$$

However, on a Riemann surface $j$ is in fact holomorphic since both $j$ and $\mathcal{J}^\nabla$ are integrable at the same time.

We shall now examine what happens when we require $(N,J^N)$ to be an almost-complex manifold, with respect to which the map $\psi$ is a holomorphic map into the twistor space. We drop here the adjective 'pseudo' but no $\mathbb{C}$-analycity is assumed throughout the text if not explicitly mentioned.

**Theorem 2.5.** *The following conditions are equivalent:*
*(i)* $\psi$ *is* $(J^N,\mathcal{J}^\nabla)$*-holomorphic*
*(ii) (α)* $\mathrm{d}f\circ J^N=\Phi^\psi\circ\mathrm{d}f$ *and*
*(β)* $f^*\nabla_u\Phi^\psi(\psi^+)=0,\ \ \forall u\in T^+N$
*(iii) (α)* $\mathrm{d}f(T^+N)\subset\psi^+$ *and*
*(β)* $f^*\nabla_u\Gamma\psi^+\subset\Gamma\psi^+,\ \ \forall u\in T^+N.$

*Proof.* This is again just like in [22]. $\psi$ is holomorphic iff

$$\mathrm{d}\psi\circ J^N=\mathcal{J}^\nabla_\psi\circ\mathrm{d}\psi.$$

Splitting into horizontal and vertical parts as in proposition 2.11 we find

$$\mathrm{d}\pi\circ\mathrm{d}\psi\circ J^N=\Phi_\psi\,\mathrm{d}\pi\circ\mathrm{d}\psi,$$

which is equivalent to *(ii α)*, and, with $X\in TN$, we have

$$P(\mathrm{d}\psi(J^NX))=\mathcal{J}^v_\psi P(\mathrm{d}\psi X). \tag{2.7}$$

(2.7) is equivalently and successively transformed into the following equations:

$$\begin{aligned}\psi^*P(J^NX)&=\psi^*\Phi\,\psi^*P(X)\\ -\psi^*\Phi\,\psi^*(\Phi P)(J^NX)&=\psi^*(\Phi P)(X)\\ -\Phi^\psi(\psi^*\pi^*\nabla)_{J^NX}\psi^*\Phi&=(\psi^*\pi^*\nabla)_X\psi^*\Phi\\ \Phi^\psi(f^*\nabla)_{J^NX}\Phi^\psi&=-(f^*\nabla)_X\Phi^\psi.\end{aligned}$$

For $u\in T^+N$ in the place of $X$ this means

$$(1+i\Phi^\psi)f^*\nabla_u\Phi^\psi=0.$$

But, as with all complex structures, we have $\Phi^\psi(f^*\nabla\Phi^\psi)=-(f^*\nabla\Phi^\psi)\Phi^\psi$ and hence we see our equation is equivalent to the present *(ii β)*. Since for $\overline{u}\in T^-N$ we get the conjugate equation, thus nothing new, we conclude (2.7) is equivalent to *(ii β)*.

*(iii)* is another restatement of the above. In particular, parts *(β)* were proved equivalent in proposition 1.10 *(iii *)*. ■

**Example 1.** We consider again a section $j$ of the bundle $\mathcal{J}(M,\omega,*)$. Since $\Phi^j = j$, *(iii α)* is immediately satisfied. Clearly $j$ is $(j,\mathcal{J}^\nabla)$-holomorphic if and only if $j$ satisfies *condition (B)* — see definition in (1.12), section 1.3.

Furthermore, one should also show that $j$ will satisfy condition *(A)* iff it is $(j,\mathcal{J}_2^\nabla)$-holomorphic. This example has already been explored as a proposition in [23], proved by different means and in the riemannian case. It explains the importance of conditions *(A)* and *(B)* for twistor theory.

**Example 2.** If we have two sections $j_1, j_2$ as above and assume one is a holomorphic map into the twistor space with respect to the other, then they are the same. The proof is obvious from the theorem.

**Example 3.** For any fixed section $j_0$, fibre preserving smooth involutions of $\mathcal{J}(M,\omega,*)$ like $j \mapsto -j_{0x}\, j\, j_{0x}$, where $j \in \pi^{-1}(x)$, or likewise $j \mapsto j\, j_0\, j\, j_0\, j$, are never holomorphic.

**Example 4.** Notice that for any section $J \in \Gamma\mathcal{J}(M)$ the bundle projection $\pi$, a map *from* the twistor space, is never $(\mathcal{J}^\nabla, J)$-holomorphic by definition of $\mathcal{J}^\nabla$.

Considering example 3 in the previous section we have found that a section $j(z) = (z, w(z))$ of $\mathcal{J}(P^1(\mathbb{C}),\omega,0)$, thus in the described coordinates, is holomorphic if and only if $w$ satisfies the differential equation

$$\frac{\partial w}{\partial \overline{z}} + w\frac{\partial w}{\partial z} - \frac{2w(w\overline{z}-z)}{1+|z|^2} = 0.$$

Notice $w = 0$ is a solution, the canonical complex structure. The reader may see the deduction of the equation in comparing the proof of the following result and the systems given in proposition 2.10 and formula (2.6). The analogy is trivial.

Let now $\alpha,\beta \in \mathrm{C}^{\infty}_{\mathbb{R}^2}(\mathbb{C})$ describe a symplectic connection on $\mathbb{R}^2$ just like in example 1 of the examples section, so that we have an associated twistor space with real coordinates $(z,w)$. Let

$$\mathcal{P}(w) = \overline{\beta} - 3\overline{\alpha}w + 3\alpha w^2 - \beta w^3.$$

**Proposition 2.12.** *A section $j(z) = (z, w(z))$ of $\mathcal{J}(\mathbb{R}^2,\omega,0)$ is holomorphic iff $w$ satisfies*

$$\frac{\partial w}{\partial \overline{z}} + w\frac{\partial w}{\partial z} + \mathcal{P}(w) = 0.$$

*Proof.* To find this we do not use the theorem before. It is wiser to consider holomorphic functions $f$ on the twistor space, thus satisfying the system in proposition 2.10, and then claim that $j$ is $(j,\mathcal{J}^{\nabla})$-holomorphic if and only if $f\circ j$ is holomorphic, $\forall f$. This corresponds to

$$\mathrm{d}(f\circ j)\left(\frac{\partial}{\partial \overline{z}} + w(z)\frac{\partial}{\partial z}\right) = 0.$$

Equivalently,

$$\frac{\partial f}{\partial \overline{z}} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial \overline{z}} + \frac{\partial f}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial \overline{z}} + w\frac{\partial f}{\partial z} + w\frac{\partial f}{\partial w}\frac{\partial w}{\partial z} + w\frac{\partial f}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial z} \;=\; 0$$

or

$$\left(\mathcal{P}(w) + \frac{\partial w}{\partial \overline{z}} + w\frac{\partial w}{\partial z}\right)\frac{\partial f}{\partial w} \;=\; 0.$$

Since there exist sufficient holomorphic functions, we are finished. ∎

In the same spirit one finds the equation for parallel sections.

# Chapter 3

## 3.1 A complex map

Throughout the present chapter we shall only refer to the twistor space $\mathcal{J}(M,\omega,0)$ — which we simply denote by $\mathcal{J}(M,\omega)$ — since most results will remain true with minor changes in the text, as the reader may care to notice.

Let us first see another proof of proposition 2.7 in a particular case. Let $(X,\omega,J,\nabla)$ be a kähler manifold, where $\nabla$ is the Levi-Civita connection of $g_{_J} = \omega(\ ,J\ )$. Then we may construct "little-twistors" inside $\mathcal{J}(X,\omega)$ which will be almost-complex submanifolds of the twistor space of $X$.

Consider an embedded holomorphic submanifold $C$ in $X$. The real tangent space $TC$ is then a complex or $J$-invariant, and hence symplectic, subbundle of $TX_{|C}$. One can also see that $TC^\omega$ is complex. What we have called a little-twistor is just $\mathcal{J}(C,\omega)$ built with the restriction of the symplectic form $\omega$ on $C$, or rather its image under the following map $I$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{J}(C,\omega) & \xrightarrow{I} & \mathcal{J}(X,\omega) \\ \rho\downarrow & & \downarrow\pi \\ C & \xrightarrow{\iota} & X \end{array}$$

where

$$I(j) = \begin{cases} j & \text{on } TC \\ J & \text{on } TC^\omega. \end{cases}$$

It is not difficult to see that, for any $x \in C$, the Siegel domain $\rho^{-1}(x)$ embeds holomorphically in $\pi^{-1}(x)$ in the way described.

Now, recall there is a result which says we can restrict $\nabla$ to the submanifold $C$, and then project onto $TC$ and its orthogonal complement $TC^\omega$, to define two new connections (*cf.* [17]). We denote them, respectively, by $\nabla^C$ and $\nabla^\perp$. Moreover, the first is the Levi-Civita connection of $C$ and the second is also easily seen to be metric and symplectic. So we can use $\nabla^C$ to define a twistor complex structure on $\mathcal{J}(C,\omega)$.

With an extra assumption, we can guarantee immediately, as we wish to, the following result.

**Lemma 3.1.** *If $C$ is totally geodesic, then the smooth embedding $I$ is holomorphic.*

*Proof.* We thus let the inclusion $\iota$ be parallel. Recall this means $\nabla_X Y \in \Gamma TC$, for all $X, Y \in \Gamma TC$, and that it is also equivalent to $C$ being totally geodesic. In this situation, we have the identity:

$$\nabla = \nabla^C \oplus \nabla^\perp.$$

To prove the claim, notice that, since $I_*$ is holomorphic along the fibres and is just the inclusion map on the horizontal subspaces, we are left to check that $I_*\mathcal{H}^{\nabla^C} \subset \mathcal{H}^\nabla$. Writing $\Phi^C$ for the canonical section over $\mathcal{J}(C,\omega)$, we have

$$\begin{aligned} I^*\Phi_j &= \Phi_{I(j)} \\ &= j \oplus J \;=\; \Phi^C_j \oplus (\rho^* J)_j \end{aligned}$$

and therefore

$$\begin{aligned} I^*\left(\pi^*\nabla_{I_*Y}\Phi\right) &= (I^*\pi^*\nabla)_Y I^*\Phi \\ &= \rho^*\nabla_Y\left(\Phi^C \oplus \rho^* J\right) \\ &= \rho^*\nabla^C_Y\Phi^C \oplus \rho^*\nabla^\perp_Y\rho^* J \;=\; 0 \end{aligned}$$

for any $Y \in \mathcal{H}^{\nabla^C}$. Finally, recall $I^*$ is injective. ■

Requiring much further study, namely to examine the converse of the lemma, we leave here the previous constructions on little twistors. We guess they would lead us to far away — in *real* geometry. There is already enough for an application, with which we proceed.

Let $(M, \omega)$ be a symplectic manifold with a symplectic connection $\nabla$. We say $\nabla$ is "locally Kähler" if for each point of $M$ we can find a neighborhood $U$ and a parallel complex structure $J$ on $U$ compatible with $\omega$. The following is an attempt towards an intuitive view of proposition 2.7.

**Proposition 3.1.** *If $M$ admits a totally geodesic holomorphic curve and if $\nabla$ is locally Kähler, then the second twistor almost-complex structure is not integrable.*

*Proof.* Since the question is local we may assume $M$ is kählerian. Let $C$ be the germ of a holomorphic and parallel curve in $M$. If $\mathcal{J}_2^\nabla$ were integrable, then the little-twistor $\mathcal{J}(C, \omega)$ passing through $J$ would be integrable. But we know this is never true (*cf.* section 2.3, example 1). ■

The problem of finding some example of a symplectic connection which is not locally Kähler must be solved, if we aim to a new concept. This is done in section 3.5.

The map we are really interested in in this section is one which is defined naturally in twistor space theory and which, in the symplectic case, gives an interesting relation between holomorphicity and symplectic connections.

Let $(M, \omega), (M_1, \omega_1)$ be two symplectic manifolds and $\sigma : M \longrightarrow M_1$ a symplectomorphism. Then $\sigma$ induces an invertible transformation of $\mathcal{J}(M, \omega) = Z$ onto $\mathcal{J}(M_1, \omega_1) = Z_1$ preserving the fibrations $\pi : Z \longrightarrow M$, *i.e.* a map $\Sigma$ such that

$$\begin{array}{ccc} Z & \xrightarrow{\Sigma} & Z_1 \\ \pi \downarrow & & \downarrow \pi_1 \\ M & \xrightarrow{\sigma} & M_1 \end{array}$$

is commutative. Indeed, for any $y \in M_1,\ j \in \pi^{-1}(\sigma^{-1}(y))$ we define

$$\Sigma(j) = \mathrm{d}\sigma \circ j \circ \mathrm{d}\sigma^{-1},$$

which is in $\pi_1^{-1}(y)$. It is trivial to check $\Sigma$ is well defined and invertible (more generally we deduce the invariance of the signatures of $g_{\Sigma(j)}$ and $g_j = \sigma^* g_{\Sigma(j)}$).

Assume $Z, Z_1$ have twistor almost-complex structures $\mathcal{J}^\nabla$ and $\mathcal{J}^{\nabla^1}$, respectively, where $\nabla^1 = \sigma\cdot\nabla$ and $\nabla$ is symplectic.

**Theorem 3.1.** *$\Sigma$ is holomorphic.*

*Proof.* Before we start to justify the conditions for $\mathbb{C}$-analycity in theorem 2.5 notice that $\Sigma$ preserves the fibres and extends to a linear map between $\operatorname{End} T_{\sigma^{-1}(y)}M$ and $\operatorname{End} T_y M_1$. Hence

$$\begin{aligned} \mathrm{d}\Sigma(jA) &= \Sigma(jA) \\ &= \Sigma(j)\Sigma(A) \;=\; \Sigma(j)\,\mathrm{d}\Sigma(A) \end{aligned}$$

and we may conclude that our map is *vertically* holomorphic.

Now conditions *(ii $\alpha,\beta$)* in the referred theorem are slightly easier to check since we just have to consider horizontal vectors over $Z = \mathcal{J}(M,\omega)$. The map $f$ there is $\sigma\circ\pi$ here, so

$$\begin{aligned} \mathrm{d}(\sigma\circ\pi)\,\mathcal{J}^h_j &= \mathrm{d}\sigma\circ j\circ \mathrm{d}\sigma^{-1}\circ\mathrm{d}\sigma\circ\mathrm{d}\pi \\ &= \Sigma(j)\circ\mathrm{d}(\sigma\circ\pi) \\ &= \Sigma^*\Phi\,\mathrm{d}(\sigma\circ\pi)_j \end{aligned}$$

proves *(ii $\alpha$)*. We recover from formula (2.7) that *(ii $\beta$)* is equivalent to a more recognisable expression of a holomorphic map:

$$(\Sigma^* P(u))^- = 0, \qquad \forall u\in\mathcal{H}^{\nabla +}$$

where $P$ is the usual projection inside $TZ_1$. Obviously the latter is satisfied *if* $\Sigma_*\mathcal{H}^\nabla = \mathcal{H}^{\nabla^1}$. This will be exactly the case, as we shall see next, when we consider the particular connection $\nabla^1$.

Let $\dim M = \dim M_1 = 2n$ and fix a real symplectic vector space $V$. Let $F, F_1$ be respectively the symplectic frame bundles of $M$ and $M_1$. Consider the $Sp(2n,\mathbb{R})$-equivariant map

$$\begin{aligned} \Lambda\;:\;& F\longrightarrow F_1 \\ & p\longmapsto \mathrm{d}\sigma\circ p \end{aligned}$$

where the points $p : V \longrightarrow T_xM$ are linear isomorphisms. If $s : U \to F$ is a section around $x \in M$, then

$$s_1 = \Lambda \circ s \circ \sigma^{-1} : \sigma(U) \longrightarrow F_1$$

is a section around $\sigma(x)$. We wish to show first that $\Lambda$ preserves the horizontal distributions induced by the connections. Let $\alpha, \alpha_1$ denote the connection 1-forms on $F$ and $F_1$.

$$\nabla_{X_x} s = s(s^*\alpha)X_x$$

and

$$\begin{aligned}(\sigma\cdot\nabla)_{Y_{\sigma(x)}} s_1 &= s_1(s_1^*\alpha_1)Y_{\sigma(x)}\\ &= \Lambda\circ s\circ\sigma^{-1}_{\sigma(x)}\left[(\Lambda\circ s\circ\sigma^{-1})^*\alpha_1\right](Y_{\sigma(x)})\\ &= \mathrm{d}\sigma\, s(s^*\Lambda^*\alpha_1)\,\mathrm{d}\sigma^{-1}(Y_{\sigma(x)})\\ &= \mathrm{d}\sigma\, s(s^*\Lambda^*\alpha_1)(\sigma^{-1}\cdot Y)_x.\end{aligned}$$

On the other hand, since $(\sigma^{-1}\cdot s_1)_x = \mathrm{d}\sigma^{-1}(s_{1\sigma(x)}) = s_x$, we have

$$\begin{aligned}(\sigma\cdot\nabla)_{Y_{\sigma(x)}} s_1 &= \sigma\cdot\left(\nabla_{\sigma^{-1}.Y}\,\sigma^{-1}\cdot s_1\right)_{\sigma(x)}\\ &= \mathrm{d}\sigma\left(\nabla_{(\sigma^{-1}.Y)_x}\, s\right)\\ &= \mathrm{d}\sigma\, s(s^*\alpha)(\sigma^{-1}\cdot Y)_x.\end{aligned}$$

Henceforth $s^*\Lambda^*\alpha_1 = s^*\alpha$ and we prove the claim that $\ker\alpha_1 = \Lambda_*\ker\alpha$ taking horizontal frames along paths in $M$ passing through $x$ (with vertical fundamental vector fields one can actually see further that $\Lambda^*\alpha_1 = \alpha$).

Finally let $\zeta : F \to Z$ be the once introduced fibre bundle with bundle map

$$\zeta(p) = pJ_0p^{-1},$$

where $J_0$ is some *positive* compatible complex structure of $V$ — *cf.* section 2.2, formula (2.3). Clearly

$$\Sigma\circ\zeta(p) = \mathrm{d}\sigma\, pJ_0p^{-1}\mathrm{d}\sigma^{-1} = \zeta_1\circ\Lambda(p)$$

and we know the $\zeta$ preserve the horizontal tangent bundles:

$$\zeta_*\ker\alpha = \mathcal{H}^\nabla, \qquad \zeta_{1*}\ker\alpha_1 = \mathcal{H}^{\sigma\cdot\nabla}.$$

Now it is no longer difficult to see that $\Sigma_*\mathcal{H}^\nabla = \mathcal{H}^{\sigma\cdot\nabla}$. ■

The reader may notice that all the constructions and results above are still valid in the general twistor space $\mathcal{J}(M)$ case. Indeed, already the proof of theorem 2.5 did not mention any particular feature of symplectic manifolds.

**Remark** An application of the last theorem is the result at the end of section 2.3. The theorem confirms what one can effectively see by computations, *i.e.* that the PDE system given there is preserved under the change of affine coordinates in $P^1(\mathbb{C})$. It also applys in the following strictly real situation: since $(\mathbb{R}^2, \omega)$ is symplectomorphic to the hyperbolic disk $(\mathcal{D}, \omega_1)$, where

$$\omega_1 = \frac{i}{2} \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{(1 - |z|^2)^2},$$

we can study $\mathcal{J}(\mathcal{D}, \omega_1)$ using the theorem and example 1 in section 2.3 (it corresponds to find the Darboux coordinates in $\mathcal{D}$ and the respective connection's parameters).

There is a partial converse to the theorem, which is only valid in the symplectic case. In the following we assume all the previous setting.

**Corollary 3.1.** *Let $\nabla^2$ be any symplectic connection on $M_1$ and suppose $\Sigma : (Z, \mathcal{J}^\nabla) \to (Z_1, \mathcal{J}^{\nabla^2})$ is holomorphic. Then*

$$\nabla^2 = \sigma \cdot \nabla$$

*i.e. $\nabla^2$ is in the affine transformation orbit of $\nabla$.*

*Proof.* We have

$$\mathcal{J}^{\nabla^2} = \mathrm{d}\Sigma \circ \mathcal{J}^\nabla \circ \mathrm{d}\Sigma^{-1} = \mathcal{J}^{\sigma \cdot \nabla}$$

so the result follows by theorem 2.4. ∎

We remark that the theorem has the apparent merit of transforming a $2^{\text{nd}}$ order PDE's problem into a $1^{\text{st}}$ order one.

One can ask which holomorphic maps $\Sigma : Z \to Z$ arise from a symplectomorphism on the base space $M$. In particular, referring to the problem raised in section 1.4, is there any *flat* symplectic connection in $\mathbb{R}^2$ for which the twistor space is not biholomorphic to $\mathbb{C} \times \mathcal{D}$, like the one associated to $\nabla^0 = \mathrm{d}$ is?

## 3.2 Kählerian twistor spaces

In order to introduce a new structure on the twistor space we need a further amount of theory from [20]. Recall the exact sequence

$$0 \longrightarrow \mathcal{V} \longrightarrow T\mathcal{J}(M,\omega) \xrightarrow{\mathrm{d}\pi} E \longrightarrow 0$$

where $E = \pi^*TM$. Also important to recall here is that, from section 2.2, we have

$$T\mathcal{J}(M,\omega) = F^sM \times_{U(n)} \mathfrak{n}_{_J} \oplus \mathbb{R}^{2n}$$

with $F^sM$ the symplectic frame bundle, $J$ some fixed element of $J(\mathbb{R}^{2n},\omega,0) = Sp(2n,\mathbb{R})/U(n)$ and $\mathfrak{n}_{_J} = \{A \in \mathfrak{sp}(2n,\omega) : \ AJ = -JA\}$.

Let $\nabla$ be a symplectic linear connection on the given $2n$-dimensional symplectic manifold. Consider again the canonical section $\Phi \in \Gamma(\mathcal{J}(M,\omega); \mathrm{End}\, E)$ and the projection $P \in \mathrm{A}^1(\mathcal{V})$ with kernel $\mathcal{H}^\nabla$ induced by the connection, which via the identity

$$\mathcal{V}_j = \left\{A \in \mathfrak{sp}(E_j, \pi^{-1}\omega) : \ \ A\Phi_j = -\Phi_j A\right\},$$

can also be seen as an endomorphism-valued 1-form on the twistor space. We can define a new connection on $E$ by

$$D = \pi^*\nabla - P,$$

which turns $\pi^*\nabla\Phi = [P,\Phi]$ equivalent to

$$D\Phi = 0.$$

It follows that $D$ on $\mathrm{End}\, E$ preserves $\mathcal{V}$ and hence $D\mathcal{J}^v = 0$. Indeed, this connection is symplectic because its difference to an obviously symplectic connection $\pi^*\nabla$ stays within $\mathfrak{sp}(E,\pi^{-1}\omega)$, and hence, as a derivation, acts trivially on the 2-form.

The isomorphism $\pi_* : \mathcal{H}^\nabla \to E$ allows us to transfer $D$, in order to give place to a new connection $D$ on $\mathcal{H}^\nabla$ satisfying

$$\begin{aligned}(D\mathcal{J}^h)X &= \pi_*^{-1}\left(D(\pi_*\mathcal{J}^hX)\right) - \mathcal{J}^h\pi_*^{-1}\left(D\pi_*X\right)\\ &= \pi_*^{-1}\left(D\Phi\right)\pi_*X \ = \ 0.\end{aligned}$$

Henceforth we have defined a $\mathbb{C}$-linear connection on $T\mathcal{J}(M,\omega) = \mathcal{V}\oplus\mathcal{H}^\nabla$ preserving this splitting, exactly in the same lines of the general twistor theory. Since $\pi_*$ resulted in a parallel and $\mathbb{C}$-linear isomorphism, one frequently identifies throughout the text corresponding object and image in $\mathcal{H}^\nabla$ and $E$.

We need the following theorem from [20] valid in general in $\mathcal{J}(M)$ and which we improved in a little detail. Thus we keep the notation referring to the symplecic case and present our proof.

**Theorem 3.2.** *The connection $D$ on the tangent bundle of $\mathcal{J}(M,\omega)$ has torsion whose vertical part is the projection of*

$$\pi^* R^\nabla$$

*into $\mathcal{V}$, and whose horizontal part is*

$$-P\wedge \mathrm{d}\pi.$$

*(This part is $\pi^*T^\nabla - P\wedge\mathrm{d}\pi$ in a more general setting.)*

*Proof.* In the following we ask the reader to distinguish the Lie bracket from the commutator bracket when appropriate. Since $P$ and $\Phi$ are parallel,

$$\begin{aligned}
[PT^D(X,Y),\Phi] &= [D_X PY - D_Y PX - P[X,Y],\Phi] \\
&= D_X[PY,\Phi] - D_Y[PX,\Phi] - [P[X,Y],\Phi] \\
&= \pi^*\nabla_X\pi^*\nabla_Y\Phi - [PX,[PY,\Phi]] \\
&\qquad -\pi^*\nabla_Y\pi^*\nabla_X\Phi + [PY,[PX,\Phi]] - \pi^*\nabla_{[X,Y]}\Phi \\
&= [\pi^*R(\pi_*X,\pi_*Y),\Phi] - [[PX,PY],\Phi]\,.
\end{aligned}$$

Since $\operatorname{ad}\Phi$ is injective on $\mathcal{V}$ and since $[\mathfrak{n}_J,\mathfrak{n}_J]\subset\mathfrak{gl}(2n,J)$, *cf.* section 2.1, we may conclude the vertical part of $T^D$ is just $P(\pi^*R^\nabla)$. The second part of the theorem is just like in [20] and uses the same procedure so we end the proof here. ∎

The present section is devoted to the study of a natural symplectic structure on $\mathcal{J}(M,\omega)$, whose analogous construction in the riemannian version of the theory turned out to be already known — see [22]. To see which twistor spaces of that kind over a 4-manifold admit a Kähler metric one might also want to have a look at [15].

Recall that $Sp(2n,\mathbb{R})/U(n)$ is a hermitian symmetric space, hence kählerian. With the help of the Killing form and a Cartan's decomposition of $\mathfrak{sp}(2n,\mathbb{R}) = \mathfrak{u}^J \oplus \mathfrak{n}_{_J}$ one defines a symplectic form on $\mathcal{J}(M,\omega)$ by

$$\Omega^\nabla \;=\; t\,\pi^*\omega - \tau,$$

where $t \in ]0,+\infty[$ is fixed and

$$\tau(X,Y) \;=\; \frac{1}{2}\mathrm{Tr}\,(PX)\Phi(PY).$$

The following has a trivial proof.

**Lemma 3.2.** $\Omega^\nabla$ *is non-degenerate and* $\mathcal{J}^\nabla$ *is compatible with it. The induced metric is positive definite.*

Although the parameter $t$ will not teach us anything special about the twistor space, besides that it could also give a pseudo-metric, we will keep it in sight since it could become important at some moment.

**Proposition 3.2.** *For any* $X,Y,Z \in T\mathcal{J}(M,\omega)$

$$\mathrm{d}\tau(X,Y,Z) \;=\; -\frac{1}{4}\mathrm{Tr}\,\left(R^{\pi^*\nabla}_{X,Y}\circ\pi^*\nabla_{Z}\Phi + R^{\pi^*\nabla}_{Y,Z}\circ\pi^*\nabla_{X}\Phi + R^{\pi^*\nabla}_{Z,X}\circ\pi^*\nabla_{Y}\Phi\right).$$

*Proof.* Let us first see $D$ is symplectic (though not torsion free). Since $D\pi^{-1}\omega = 0$ on $E$ we are left to check $D\tau = 0$.

$$\begin{aligned}
D_X\tau\,(Y,Z) &= X(\tau(Y,Z)) - \tau(D_XY,Z) - \tau(Y,D_XZ)\\
&= X(\tau(Y,Z)) - \tfrac{1}{2}\mathrm{Tr}\,\left(P(D_XY)\Phi PZ + PY\Phi P(D_XZ)\right)\\
&= X(\tau(Y,Z)) - \tfrac{1}{2}\mathrm{Tr}\,D_X(PY\,\Phi\,PZ)\\
&= X(\tau(Y,Z)) - \mathrm{d}\left(\tfrac{1}{2}\mathrm{Tr}\,(PY\,\Phi\,PZ)\right)(X) \;=\; 0
\end{aligned}$$

by lemma 1.2, where we have seen the vertical vector fields as elements of $\mathrm{A}^0(\mathrm{End}\,E)$. Now we use proposition 1.8 to find that

$$\begin{aligned}
\mathrm{d}\tau(X,Y,Z) &= \mathrm{d}\left(\mathrm{Tr}\,(\tau\,\tfrac{1}{2}\mathrm{Id}\wedge\mathrm{Id})\right)(X,Y,Z)\\
&= \tau(T^D_{X,Y},Z) + \tau(T^D_{Y,Z},X) + \tau(T^D_{Z,X},Y).
\end{aligned}$$

Since

$$\begin{aligned}\tau(T^D_{X,Y}, Z) &= \frac{1}{4}\mathrm{Tr}\left([PT^D_{X,Y}, \Phi]PZ\right)\\ &= -\frac{1}{4}\mathrm{Tr}\left(\pi^* R^{\nabla}_{\pi_* X, \pi_* Y}[PZ, \Phi]\right)\\ &= -\frac{1}{4}\mathrm{Tr}\left(R^{\pi^*\nabla}_{X,Y} \circ \pi^*\nabla_Z \Phi\right)\end{aligned}$$

we are finished with the proof. ■

**Theorem 3.3.** $\Omega^{\nabla}$ *is closed if and only if* $\nabla$ *is flat. In such case,* $\mathcal{J}(M,\omega)$ *is a Kähler manifold.*

*Proof.* Since $\mathrm{d}\pi^*\omega = 0$, we only have to do an analysis on $\mathrm{d}\tau$ of four cases with three horizontal or vertical tangent vectors $X, Y, Z$.

The only possible non-trivial case is say $X, Y$ horizontal and $Z$ vertical. Since $\tau$ on $\mathcal{V}$ is non-degenerate, $\mathrm{d}\tau(X,Y,Z) = \tau(T^D_{X,Y}, Z) = 0$ for all those $X, Y, Z$ iff $P(T^D) = 0$. Equivalently, $[\pi^* R^{\nabla}, \Phi] = 0$, or

$$[R^{\nabla}_x, j] = 0, \qquad \forall j \in \pi^{-1}(x),\ \ x \in M.$$

A bit of work with the matrices presented in proposition 2.3 yields $R = 0$. Not happy with that, we found the following application of representation theory leading to the same conclusion. As before, for any compatible $J$ with $(\mathbb{R}^{2n}, \omega)$, let $\mathfrak{u}^J$ be the unitary Lie algebra $\mathfrak{sp}(2n, \mathbb{R}) \cap \mathfrak{gl}(2n, J)$. It is then trivial to see that

$$\mathfrak{h} = \bigcap_{J \in J(2n,\omega,0)} \mathfrak{u}^J$$

is a $Sp(2n,\omega)$-module under the action $g \cdot R = gRg^{-1}$. Because $\mathfrak{sp}(2n,\omega)$ is irreducible, we have $\mathfrak{h} = 0$ and thus the 'only if' part of the theorem.

For the last conclusion we just recall that $R^{\nabla} = 0$ implies integrability of the almost-complex structure $\mathcal{J}^{\nabla}$ as well. ■

Suppose the connection is flat. Then how come $D$ is hermitian, the twistor space is Kähler and $T^D$ is not 0? This 'ill-posed question' is immediately clarified by proposition 1.9, since $P\wedge \mathrm{d}\pi$ is not type (2,0). As one may care to check, this comes from the fact that $P\,\Phi=-\Phi\,P$.

Let $\langle\ ,\ \rangle$ be the induced metric, so that

$$\langle X,Y\rangle \;=\; t\,\pi^*\omega(X,\mathcal{J}^\nabla Y)+\frac{1}{2}\mathrm{Tr}\,(PXPY)$$

and thus $\mathcal{H}^\nabla\perp\mathcal{V}$. Let $\cdot^v$ denote the vertical part of any tangent-valued tensor.

**Theorem 3.4.** *(i) The Levi-Civita connection of $\langle\ ,\ \rangle$ is given by*

$$D^g_{X}Y \;=\; D_{X}Y-PY(\pi_*X)-\frac{1}{2}\pi^*R^{\;v}_{X,Y}+S(X,Y)$$

*where $S$ is symmetric and defined both by*

$$\langle S^v(X,Y),A\rangle \;=\; \langle A\pi_*X,\pi_*Y\rangle,\qquad \forall A\in\mathcal{V},$$

*and*

$$\langle S^h(X,B),Y\rangle \;=\; \frac{1}{2}\langle \pi^*R^{\;v}_{X,Y},B\rangle,\qquad \forall Y\in\mathcal{H}^\nabla.$$

*Hence for $X,Y\in\mathcal{H}^\nabla$ and $A,B\in\mathcal{V}$ we have*

$$S^v(X,A)=S^v(A,B)=0,$$
$$S^h(X,Y)=S^h(A,B)=0.$$

*(ii) The fibres $\pi^{-1}(x)$, $x\in M$, are totally-geodesic in $\mathcal{J}(M,\omega)$.*
*(iii) If $\nabla$ is flat, then $D^g\mathcal{J}^\nabla=0$.*

*Proof.* *(i)* Note that $S^h$ is symmetric by definition and that, to see $S^v$ is symmetric, we just have to check every $A\in\mathcal{V}$ is self-adjoint:

$$\begin{aligned}\langle A\pi_*X,\pi_*Y\rangle \;&=\; t\,\omega(A\pi_*X,\Phi\pi_*Y)\\ &=\; t\,\omega(\pi_*X,\Phi A\pi_*Y)\;=\;\langle\pi_*X,A\pi_*Y\rangle.\end{aligned}$$

Now let us see the torsion condition:

$$\begin{aligned} T^{D^g}(X,Y) &= T^D(X,Y) - PY(\pi_*X) - \frac{1}{2}\pi^*R^{\ v}_{X,Y} + S(X,Y) \\ &\qquad +PX(\pi_*Y) + \frac{1}{2}\pi^*R^{\ v}_{Y,X} - S(Y,X) \\ &= T^D(X,Y) + P\wedge \mathrm{d}\pi(X,Y) - \pi^*R^{\ v}_{X,Y} \ = \ 0. \end{aligned}$$

For the metric condition it is easier to let, from now on, $X, Y, Z$ denote horizontal and $A, B, C$ vertical tangent vectors. We already know $D$ is hermitian, so to simplify computations let $\xi = D^g - D$. Then

$$\begin{aligned} \xi_X Y &= -\frac{1}{2}\pi^*R^{\ v}_{X,Y} + S^v(X,Y), & \xi_X A &= -AX + S^h(X,A), \\ \xi_A X &= S^h(X,A), & \xi_A B &= 0 \end{aligned}$$

and thus in particular, from the last formula, we deduce *(ii)*. Now

$$D^g_X\langle\ ,\ \rangle(Y,Z) \ = \ -\langle \xi_X Y, Z\rangle - \langle Y, \xi_X Z\rangle \ = \ 0,$$

$$\begin{aligned} D^g_X\langle\ ,\ \rangle(Y,A) &= -\langle \xi_X Y, A\rangle - \langle Y, \xi_X A\rangle \\ &= \frac{1}{2}\langle \pi^*R^{\ v}_{X,Y}, A\rangle - \langle S^v(X,Y), A\rangle \\ &\qquad + \langle Y, AX\rangle - \langle Y, S^h(X,A)\rangle \ = \ 0, \end{aligned}$$

$$-D^g_X\langle\ ,\ \rangle(A,B) \ = \ \langle \xi_X A, B\rangle + \langle A, \xi_X B\rangle \ = \ 0,$$

$$\begin{aligned} -D^g_A\langle\ ,\ \rangle(X,Y) &= \langle S^h(X,A), Y\rangle + \langle X, S^h(Y,A)\rangle \\ &= \frac{1}{2}\langle \pi^*R^{\ v}_{X,Y}, A\rangle + \frac{1}{2}\langle \pi^*R^{\ v}_{Y,X}, A\rangle \ = \ 0, \\ -D^g_A\langle\ ,\ \rangle(X,B) &= \langle \xi_A X, B\rangle + \langle X, \xi_A B\rangle \ = \ 0, \end{aligned}$$

and finally

$$-D^g_A\langle\ ,\ \rangle(B,C) \ = \ 0.$$

*(iii)* Of course we know this immediately from theorems 1.3 and 3.3. But we would like to confirm: if $\nabla$ is flat then $S^h = 0$ too. Hence for *all* vector fields

$$D^g_X \mathcal{J}^\nabla Y \ = \ \mathcal{J}^\nabla D_X Y - \mathcal{J}^\nabla PY(\pi_*X) + S^v(X, \mathcal{J}^\nabla Y).$$

However, $\langle S^v(X,\mathcal{J}^\nabla Y), A\rangle = \langle \mathcal{J}^\nabla S^v(X,Y), A\rangle$ is an easy computation and valid in general, so we are finished. ■

Notice that the proof of *(iii)* shows the coherence of our results with the well known theory of the second fundamental form in Kähler geometry. Furthermore, one can write more explicitly

$$S^v_j(X,Y) \;=\; -\frac{t}{2}\Big\{\omega(X,\ )jY+\omega(jY,\ )X+\omega(jX,\ )Y+\omega(Y,\ )jX\Big\}$$

and also construct a symplectic-orthonormal basis of $\mathcal{V}$ induced by a given such basis on $\mathcal{H}^\nabla$.

An interesting question still remains: assuming $\nabla$ is complete, is the same true for $D^g$?

Next we present some of the results we have found about the kählerian twistor space $\mathcal{J}(M,\omega)$. Since they are not used anymore we do not show their long proofs. Untill the end of the section assume $R^\nabla=0$.

**Theorem 3.5.** *Let* $\Pi$ *be a 2-plane in* $T_j\mathcal{J}(M,\omega)$ *spanned by the orthonormal basis* $\{X+A,Y+B\},\ X,Y\in\mathcal{H}^\nabla,\ A,B\in\mathcal{V}$. *Then the sectional curvature of* $\Pi$ *is*

$$\begin{aligned} k_j(\Pi) &= -\langle R^{D^g}(X+A,Y+B)(X+A),Y+B\rangle \\ &= \frac{t^2}{2}\Big(\|X\|_1^2\|Y\|_1^2+3\omega(X,Y)^2-\langle X,Y\rangle_1^2\Big) \\ &\quad -t\|BX-AY\|_1^2-2t\langle[B,A]X,Y\rangle_1-\|[B,A]\|^2 \end{aligned}$$

*where* $\langle\ ,\ \rangle_1=\omega(\ ,j\ )$, *the case* $t=1$, *and* $[\,,\,]$ *is the commutator. Thus*

$$k_j(\Pi)\ \begin{cases} >0 & \text{for } \Pi\subset\mathcal{H}^\nabla \\ <0 & \text{for } \Pi\subset\mathcal{V}. \end{cases}$$

We remark that the second part of the theorem can *also* be obtained from Gauss's equations. First the reader may recall from the theory in section 1.2 that the horizontal distribution is integrable when $\nabla$ is flat (*cf.* section 3.5). Then, the horizontal leaves are quite immediately seen to have $\pi^*\nabla$ for Levi-Civita connection with the *induced* metric, and hence to be flat. Finally, for $X,Y$ horizontal and orthonormal, the formula of Gauss says

$$\begin{aligned} k_j\{X,Y\} &= \|S(X,Y)\|^2 - \langle S(X,X), S(Y,Y)\rangle \\ &= \langle S(X,Y)X,Y\rangle - \langle S(X,X)Y,Y\rangle \;=\; \text{etc} \end{aligned}$$

and the claim follows by Cauchy's inequality. For the totally geodesic fibres of $\mathcal{J}(M,\omega)$ we recall that $-\|[B,A]\|^2$ is the sectional curvature of the hyperbolic space $Sp(2n,\mathbb{R})/U(n)$.

As suggested by theorem 3.4 one can try to find the Cauchy-Riemann operator on the tangent bundle of $\mathcal{J}(M,\omega)$. We will proceed to do this in the kählerian case, hoping to make some starting point on the understanding of the former.

**Proposition 3.3.** *(i) A vector field $Y\in\mathfrak{X}_{\mathcal{J}(M,\omega)}$ is holomorphic iff*

$$D_XY + \mathcal{J}^\nabla D_{\mathcal{J}^\nabla X}Y - 2(PY)\pi_*X \;=\; 0, \qquad \forall X.$$

*(ii) $\mathcal{H}^\nabla$ is a holomorphic subvector bundle of $T\mathcal{J}(M,\omega)$.*
*(iii) $R^D$ is a (1,1)-form.*

*Proof.* *(i)* From the theory in section 1.3, $\overline{\partial}_{\mathcal{T}} = {}'' \circ D^g$. Hence

$$\begin{aligned} \overline{\partial}_{X+i\mathcal{J}^\nabla X}(Y - i\mathcal{J}^\nabla Y) &= D^g_XY + D^g_{\mathcal{J}^\nabla X}\mathcal{J}^\nabla Y + i\left(D^g_{\mathcal{J}^\nabla X}Y - \mathcal{J}^\nabla D^g_XY\right) \\ &= D^g_XY + \mathcal{J}^\nabla D^g_{\mathcal{J}^\nabla X}Y - i\mathcal{J}^\nabla\left(D^g_XY + \mathcal{J}^\nabla D^g_{\mathcal{J}^\nabla X}Y\right). \end{aligned}$$

Therefore $\overline{\partial}$ operates as the real part of the above, which is equal to

$$\begin{aligned} & D_XY - (PY)\pi_*X + S(X,Y) + \mathcal{J}^\nabla D_{\mathcal{J}^\nabla X}Y \\ &\qquad -\mathcal{J}^\nabla(PY)\pi_*\mathcal{J}^\nabla X + \mathcal{J}^\nabla S(\mathcal{J}^\nabla X, Y) \\ =\;& D_XY + \mathcal{J}^\nabla D_{\mathcal{J}^\nabla X}Y - 2(PY)\pi_*X. \end{aligned}$$

*(ii)* There is an easy way to justify this. We have seen $D$ is a hermitian connection on $\mathcal{H}^\nabla \simeq E$. From the formula above we immediately find that $D$ determines

a $\overline{\partial}$-operator on $E$ coinciding with $\overline{\partial}_{\mathcal{T}}$, hence integrable. By Koszul-Malgrange's theorem, theorem 1.2 in this work, *(iii)* follows: $R^D$ must not have (0,2)-part.

*(iii)* This actually 'appeared' before *(ii)* and implies *(ii)*, since we computed (see section 3.4, proof of corollary 3.2)

$$R^D_{X,Y} = -[PX, PY].$$

Then recall a result from section 2.1 which says $[\mathfrak{n}^+_J, \mathfrak{n}^+_J] = 0$. ■

Notice $\mathcal{V} \subset \operatorname{End} E$ also inherits an (integrable) almost-complex structure as vector bundle by Koszul-Malgrange's result. However, this has nothing to do with $D^g$ or $\mathcal{J}^\nabla$.

In conclusion, the kählerian twistor space, *i.e.* with the metric $\langle\ ,\ \rangle$ and the hypothesis $R^\nabla = 0$, has holomorphic charts in $\mathbb{C}^n \times \mathbb{C}^{\frac{1}{2}n(n+1)}$ like

$$H \times W \qquad \text{or} \qquad U \times V$$

with $H \times \{w\}$ horizontal and $\{x\} \times V$ vertical, but never a chart of the kind $H \times V$. This is not new, it agrees with example 4 in section 2.4.

## 3.3 Holomorphic completeness and the Penrose transform

We have made an attempt to define a Penrose transform on the twistor space of a symplectic manifold. In the following considerations we just present a possible complex-differential geometric point of view. Let us start by recalling some basic results about manifolds like $\mathcal{J}(M, \omega) = Z$ (we use this notation for the moment).

In a few occasions before, we have mentioned the existence of smooth sections of our fibre bundle over $M$. A theorem of N. Steenrod and others in the $C^0$-paracompact category assures that it is possible to construct global $C^0$-sections on a bundle with fibre a cell. Furthermore, there exist global extensions of any prescribed section over a closed set which extends to a neighborhood of this set (*cf.* [14]).

The above was homotopy theory — to find the smooth section of $Z$ one must use some algebraic techniques (see [18]). In either way we conclude

$$H^p(Z,\mathbb{R}) = H^p(M,\mathbb{R})$$

and thus the vanishing of cohomology for a large number of $p$'s. The reader may actually see an explicit map in section 3.4 from which to derive a strong deformation retract from $Z$ onto a "0" section. There, we shall continue our (real) de Rham cohomology study.

The previous result, the vanishing of cohomology, has a further resemblance to a completely analogous claim in a completely holomorphic setting. The reader may deduce why in between the lines of what follows.

Let $Z$ be a complex manifold of dimension $m$. Recall that $Z$ is said to be strongly $q$-pseudoconvex if it admits a smooth exhaustion function which is strongly $q$-pseudoconvex outside of a compact subset, *i.e.* there exists $\phi : Z \to \mathbb{R}$ of class $C^\infty$ such that

$$L(\phi) : TZ \otimes TZ \longrightarrow \mathbb{R}$$

has at least $m-q+1$ positive eigenvalues in the complement of a compact subset $C$, and the level sets $\{x \in Z : \phi(x) < c\}$, $c \in \mathbb{R}$, are relatively compact in $Z$. When the set $C$ is empty one says that $N$ is $q$-complete. Recall that the Levi form

$$L(\phi) = 4\sum_{i,j} \frac{\partial^2\phi}{\partial z_i \partial \overline{z}_j} \mathrm{d}z_i \otimes \mathrm{d}\overline{z}_j$$

is a hermitian 2-tensor, independent of choice of the chart $(z_1, \dots, z_m)$ of $Z$. From the definition we have that $q$-completeness implies $q+1$-completeness. One also gives the name Stein to 1-complete manifolds.

Now we need the following theorem of H. Wu. A good reference for the proof, which we omit, is [27].

**Theorem 3.6.** *A simply connected complete Kähler manifold $X$ of everywhere nonpositive sectional curvature is a Stein manifold.*

The proof contains the following arguments. Let $\phi : X \to \mathbb{R}$ be the riemannian distance function from a fixed point $p \in X$. Then it is proved that $\phi^2$ is smooth and strictly plurisubharmonic. It is an exhaustion function due to completeness of the metric: a bounded and closed set is compact.

**Remark.** Besides $\mathbb{C}^n$, the canonical example to which Wu's theorem applies is the Siegel domain $J(\mathbb{R}^{2n}, \omega, 0)$. For example, in the 1-disk every point $z$ has distance to the origin

$$\log \frac{1+|z|}{1-|z|}.$$

Hence the square of the distance function in $Sp(2n,\mathbb{R})/U(n)$ with invariant hyperbolic metric is $\mathrm{C}^\infty$. Also we remark that the same result does not apply to all components of $J(2n, \omega, *)$, as their natural metrics may be indefinite. Yet they are Stein spaces as we proved in section 2.1.

Now let $(M, \omega, \nabla)$ be a symplectic manifold of dimension $2n$ with a symplectic connection of Ricci type, *i.e.* the Weyl part of the curvature tensor vanishes. Consider the twistor space $\mathcal{J}(M, \omega)$, a complex manifold of dimension $n+k$ where $k = n(n+1)/2 = \dim$ Siegel domain. As usual, let $\mathcal{J}^\nabla$ denote the complex structure and $\pi$ the projection to $M$.

**Lemma 3.3.** *Let $D$ be a domain in $\mathbb{C}^m$ and $X$ a regular complex analytic subspace.*

*If $\phi \in \mathrm{C}^2_D$ then*

$$L(\phi)_{|TX\otimes TX} = L(\phi_{|X}).$$

*Proof.* We know that for every $z \in X$ there is a chart $(z_1, \ldots, z_m)$ in a neighborhood $U$ of $z$ such that $X \cap U = \{z \in U : z_{k+1} = \ldots = z_m = 0\}$. Since $T_z(X \cap U) = \{u \in T_zU : \ \mathrm{d}z_i(u) = 0, \ i > k\}$ we find the result just by looking at the definition of the Levi form. ■

As we said, $M$ always admits a smooth and compatible almost-complex structure $J$, so we define a smooth function $h$ on $\mathcal{J}(M,\omega)$ to be the square of the distance in each fibre to the section $J$ — which we know to arise from a smooth riemannian metric on the vertical bundle $\ker \mathrm{d}\pi$.

**Theorem 3.7.** *If $M$ has a smooth exhaustion function $\phi$, then $\mathcal{J}(M,\omega)$ is $n+1$-complete.*

*Proof.* Let

$$\psi = h + \phi \circ \pi.$$

It is a smooth and exhaustion function. To prove this notice that $h$ is positive so the closed level sets of $\psi$ are inside the closed level sets of $\phi \circ \pi$ for the same constant $c$. These project onto a compact subset $K_c$ of $M$. Then we have that

$$\begin{aligned} \{j \in \mathcal{J}(M,\omega) : \ \psi(j) \leq c\} \quad &\subset \quad \big\{j \in \pi^{-1}(K_c) : \ \psi(j) \leq c\big\} \\ &\subset \quad \big\{j \in \pi^{-1}(K_c) : \ h(j) \leq c + \sup_{K_c} |\phi|\big\} \end{aligned}$$

and, since the biggest set is compact, the closed level sets of $\psi$ are compact.

Now, for any $x \in M$, we apply the lemma to the complex submanifold $\pi^{-1}(x)$ and use the previous theorem to find that $L(\psi_{|\pi^{-1}(x)}) = L(h_{|\pi^{-1}(x)})$ has $k$ positive eigenvalues. Since $L(\psi)$ is hermitian symmetric, there is an orthogonal complement for $\ker \mathrm{d}\pi$ and we may conclude that $L(\psi)$ has at least $k$ positive eigenvalues. Hence $\mathcal{J}(M,\omega)$ is $q$-complete, where $q$ is such that $n + k - q + 1 = k$. ■

**Example 1.** If $M$ is compact we may take $\phi = 0$ in the theorem above. In particular, $\mathcal{J}(P^n(\mathbb{C}),\omega)$ is $n+1$-complete (and not less, see below).

**Example 2.** If $M$ has some riemannian structure for which there is a pole, *i.e.* there exists $x_0 \in M$ such that $\exp : T_{x_0}M \to M$ is a diffeomorphism, then we may take $\phi = \|\exp^{-1}\|^2$.

**Example 3.** Let $M = B_\epsilon(0)$, the open ball of radius $\epsilon$ in $(\mathbb{R}^{2n}, \omega_0)$. For this case we found $\phi(x) = -\log(\epsilon^2 - \|x\|^2)$, which is the famous function of K. Oka.

One must realise now that the difficult thing is to find completeness below $n+1$. This will certainly involve the horizontal part of $\mathcal{J}^\nabla$, which so much characterises twistor spaces.

**Remark.** In general, it is impossible to find a better result than that of the theorem: we know the Levi-Civita connection of $P^n(\mathbb{C})$ is of Ricci type and we have seen that parallel complex structures embed holomorphically into the twistor space. On the other hand it is well known that a $q$-complete space does not have $n$-dimensional compact analytic submanifolds, for any $n \geq q$.

By the same token, the *kählerian* twistor space $\mathcal{J}(\mathbb{T}^{2n}, \omega)$ is just holomorphically $n+1$-complete.

However, with some restriction, it may well happen that it is possible to carry on. So far, the only indication we have of this is example 2 in the examples section combined with theorem 3.1: as we remarked there, the twistor space of $\mathbb{R}^2$ with trivial connection $\nabla^0$, and hence with all $\sigma \cdot \nabla^0$, is 1-complete or Stein.

In a parallelism with what was done in [2, 12, 21] in the celebrated riemannian case of $P^3(\mathbb{C}) \to S^4$ we finally arrive to a point where, shortly, we define “Penrose transform” to be the direct image of any complex analytic sheaf over twistor space onto the base space. Thus a functor $\mathcal{O} \to \mathrm{C}^\infty$.

**Theorem 3.8.** *Let $(M, \omega, \nabla)$ be as above and $\mathcal{F}$ a coherent analytic sheaf over $\mathcal{J}(M, \omega)$. Then*

$$R^q \pi_* \mathcal{F} = 0, \qquad \forall q \geq n+1.$$

*Proof.* Recall $R^q\pi_*\mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto H^q(\pi^{-1}U, \mathcal{F})$. Hence the stalk at $x \in M$ is

$$\lim_{U \ni x} \operatorname{ind} H^q(\pi^{-1}U, \mathcal{F}).$$

Now, for a sufficiently small neighborhood $U$ of $x$, there is a chart $\sigma : U \to B \subset \mathbb{R}^{2n}$ such that $\sigma^*\omega_0 = \omega$ and $\sigma(x) = 0$. Since we have a theorem saying there is a biholomorphism

$$\Sigma : (\mathcal{J}(U,\omega), \mathcal{J}^\nabla) \longrightarrow (\mathcal{J}(B,\omega_0), \mathcal{J}^{\sigma\cdot\nabla}),$$

we may suppose our base space is $B$ and the coherent analytic sheaf is $\Sigma_*\mathcal{F}$.

Finally, the $\{B_\epsilon(0)\}_{\epsilon>0}$ form a basis for the neighborhoods of 0 and, by example 3 above, all $\mathcal{J}(B_\epsilon, \omega_0) = \pi^{-1}(B_\epsilon)$ are n+1-complete. By definition of inductive limit we find that $(R^q\pi_*\mathcal{F})_x = 0,\ \forall q \geq n+1$, appealing to Andreotti-Grauert's "t. de finitude pour la cohomologie des espaces complexes" (*cf.* [1]). ■

Although we know $H^q(\pi^{-1}(x), \iota^*\mathcal{F}) = 0,\ \ \forall q \geq 1$, where $\iota$ is the inclusion map, one has to notice in the above proof that the $\{\pi^{-1}(U)\}$ do not form a basis of the neighborhoods of $\pi^{-1}(x)$, as they always do in the riemannian case (the fibre is compact). Another remark is that one could have used any metric on a neighborhood of $x \in M$ instead of appealing to the theorems of Darboux and the one specific to twistor spaces.

## 3.4 Further results

We show here some further results or remarks of various kinds on the twistor space of a symplectic manifold. They could have been easily dispersed along the previous sections, but then they would not add much more insight and would not call attention to possible new directions of study.

Let us continue the remarks of the last section on cohomology with constant coefficients. Again let $Z = \mathcal{J}(M,\omega)$. Since $Z$ has an almost complex structure, it is an orientable manifold. So Poincaré-duality together with contractibility yields

$$\begin{aligned} H^p_{\mathrm{c}}(Z) &\simeq \big(H^{2n+2k-p}(Z)\big)^* \\ &\simeq H^{p-2k}_{\mathrm{c}}(M) \end{aligned} \tag{3.1}$$

where $2k = n(n+1)$ is the fibre dimension. Along with this isomorphism we can also consider a *cohomology with vertical compact support* in the sense of R. Bott and L. Tu (see [5]). This was done for vector bundles, so now is the time to recall a known map — to which we had already alluded in section 3.3. Let $J_0 \in \Gamma(M;Z)$.

**Proposition 3.4.** *Let* $B(\underline{\mathfrak{n}}) = \{S \in \mathfrak{sp}(TM,\omega) : \ \|S\| < 1, \ \ J_0S = -SJ_0\}$. *Then the map*

$$\begin{aligned} f \ : \ Z &\longrightarrow B(\underline{\mathfrak{n}}) \\ J &\longmapsto (J+J_0)^{-1}(J-J_0) \end{aligned}$$

*is a well defined diffeomorphism.*

Note $\|S\|$ is the max-norm on operators induced by the metric $g_{J_0}$. The proof of the proposition, though in another context, can be found as an exercise in [4].

Henceforth $Z$ sits as an open set in a vector bundle — it simply does not have a preferred zero section — and thus we may define

$$\pi_* : H^p_{\mathrm{cv}}(Z) \longrightarrow H^{p-2k}(M)$$

by integration along the fibre of the closed $p$-forms which have compact support along each fibre on the components whose vertical part is top degree $2k$, the cv-forms, and by 0 in any other case. In [5] it is proved independence of chart and class, and that $\mathrm{d}\,\pi_* = \pi_*\,\mathrm{d}$. The only necessary condition to do such a fibre-integration, as one may easily guess why, is that $Z$ must be an oriented fibre bundle. By 'oriented fibre bundle' we mean there exists a collection of trivializations of the bundle with all transition functions preserving the orientation of a standard orientable fibre.But this is true for every $P\times_G Y$ where $Y$ is a $G$-space, $P$ is an oriented principal $G$-bundle and $G$ is connected. The proof is straightforward and fails if $G$ is not connected as the case of $S^1\times_{\{0,1\}}\mathbb{R}\to S^1$ will show.

In our case, $P$ is the obviously oriented bundle of symplectic frames and $Y=J(2n,\omega,0)$. By a theorem of [5] on the oriented vector bundle case we may conclude

$$H^p_{\mathrm{cv}}(Z)\simeq H^{p-2k}(M),$$

a Thom isomorphism which agrees with (3.1) if $M$ is compact. (A marginal question for us still holds: since $Z$ oriented bundle implies $Z$ orientable manifold, are there counter examples for the converse, assuming already the fibre and base space are orientable?)

One could also try to see what is L$^2$-cohomology (or intersection cohomology?) over $Z$ with values on a vector bundle. In particular, the latter could be assumed to be a homogeneous $G_x=Sp(T_xM,\omega_x)$-vector bundle along each fibre, *i.e.* $E\to Z$ such that

$$\begin{aligned} E_{|\,\pi^{-1}(x)} &= \frac{\pi^{-1}(x)\times E_x}{\sim} \\ &= \left\{(j,e)\sim(gjg^{-1},g\cdot e):\ g\in G_x\right\} \end{aligned}$$

and where $E_x$ is an irreducible $G_x$-module. There are sophisticated results due to A. Borel to apply on $H^i_{(2)}(\pi^{-1}(x),E_|)$. From this it would follow, for instance, the study of $G_x$-invariant forms on $Z$.

Turning now to the holomorphic structure $\mathcal{J}^\nabla$ induced on $Z$ by a Ricci type symplectic connection on $M$, there should exist some criteria to determine when is $\pi^*F$ a holomorphic vector bundle, where $\pi^*F$ is sitting in

$$\begin{array}{ccc} \pi^*F & \longrightarrow & F \\ \downarrow & & \downarrow \\ Z & \stackrel{\pi}{\longrightarrow} & M, \end{array}$$

and $F$ is a given vector bundle. This was done for the 4-dimensional riemannian case in [3]. Notice $\pi^*F$ agrees with the above construction, with trivial $G_x$ action on each $F_x,\ \ x\in M$. Other questions follow: which operator does the Cauchy-Riemann on $\pi^*F$ induce on $F$ and, starting with a holomorphic vector bundle $E\to Z$, which $R^q\pi_*\mathcal{O}(E)$ induce a vector bundle on $M$?

Now, notice the bundle $B(\underline{\mathfrak{n}})$ in proposition 3.4 is associated to the principal $U(n)$-bundle of unitary frames, so it is particularly suitable for the study of the 'symplectic' twistor space of a Kähler manifold. Indeed, as it was explained after theorem 2.2, $B(\underline{\mathfrak{n}})$ carries a twistor almost-complex structure if $\nabla J_0=0$, *i.e.* $\nabla$ is symplectic and complex linear. In this case the map $f$ in proposition 3.4 becomes holomorphic.

Until the end of this section assume $(M,\omega,J_0)$ is a Riemann surface, and let $h$ denote the correspondent hermitian structure on $T'M$ under the $\mathbb{C}$-isomorphism $J_0^+:TM\to T'M$.

**Proposition 3.5.** *$\mathcal{J}(M,\omega)=B_1(T'M\otimes_c T'M)$ up to diffeomorphism, and where $B_1$ represents the radius 1 disc bundle.*

*Proof.* Note first that if $S\in\underline{\mathfrak{n}}=\{S\in\operatorname{End}TM:\ SJ_0=-J_0S\}$, then

$$\begin{aligned}\operatorname{Tr}S &= \operatorname{Tr}J_0^{-1}SJ_0\\ &= -\operatorname{Tr}J_0SJ_0 \;=\; -\operatorname{Tr}S,\end{aligned}$$

and so $S\in\mathfrak{sl}(TM)=\mathfrak{sp}(TM,\omega)$. Hence

$$\underline{\mathfrak{n}}=\operatorname{End}_{\overline{c}}T'M=T'M\otimes_c T'M.$$

Moreover, the isomorphisms resume to $S=u\,h(v,\ )=u\otimes v$ and

$$\begin{aligned}\|S\| &= \sup_{|U|=1}|SU|\\ &= \sup|h(v,\ )||u| \;\leq\; |u||v|.\end{aligned}$$

Taking $U=v/|v|$, we find $\|S\|=|u||v|$. Now compare with proposition 3.4. ∎

(Is there a relation here to the quadratic differentials of Teichmüller's space?)

As the reader knows there are various ways to describe the Siegel domain. Hence the result above and the one that follows. Let $M$ be connected, orientable and compact. Recall that the Euler characteristic is equal to $2-2g$ where $g$ is the genus of $M$. Recall also that we have seen a way in which to embed $\mathcal{J}(M,\omega)$ in $P^1(TM\otimes\mathbb{C})=Gr(M)$, at least fiberwise (*cf.* section 2.1, proposition 2.4). Because $Gr(M)$ is associated to an even Euler number principal $U(1)$-bundle, we may use a result from [18] on the classification of sphere bundles over Riemann surfaces, to conclude that $Gr(M)$ is diffeomorphic to the trivial bundle $M\times S^2$. In other words, $M$ is parametrizing a disc 'flowing' inside $S^2$, the twistor's fibres, with boundary *the* principal bundle of unitary frames.

A last corollary to theorem 3.2 on honest twistor theory follows.

**Corollary 3.2.** *If $M$ is a Riemann surface, then $\mathcal{H}^\nabla$ and $\mathcal{V}$ are holomorphic line bundles over $\mathcal{J}(M,\omega)$.*

*Proof.* Let $D=\pi^*\nabla-P$ be the connection defined in section 3.2, which is induced by the Levi-Civita connection $\nabla$ of $M$. First we compute in any dimension

$$\begin{aligned}
\mathrm{d}^{\pi^*\nabla}P(X,Y) &= \pi^*\nabla_X(PY)-\pi^*\nabla_Y(PX)-P[X,Y]\\
&= \pi^*\nabla_X PY-\pi^*\nabla_Y PX+P\left(T^D(X,Y)-D_XY+D_YX\right)\\
&= \pi^*\nabla_X PY-\pi^*\nabla_Y PX+P(\pi^*R^\nabla_{X,Y})\\
&\qquad -\pi^*\nabla_X PY+[PX,PY]+\pi^*\nabla_Y PX-[PY,PX]\\
&= P(\pi^*R_{X,Y})+2[PX,PY].
\end{aligned}$$

Hence, from the proof of proposition 1.4, we have that

$$\begin{aligned}
R^D &= R^{\pi^*\nabla}-\mathrm{d}^{\pi^*\nabla}P+P\wedge P\\
&= \pi^*R-P(\pi^*R)-P\wedge P.
\end{aligned}$$

Now, recall the twistor space is always a complex 2-manifold and $D$ is a $\mathbb{C}$-linear connection. Moreover, in dimension $n = 1$ we also have that $R^\nabla$ is proportional to $\omega$ and so it is type (1,1) for all $j$ in any fibre of the twistor space — an assertion equivalent to $\pi^*R$ being (1,1) for $\mathcal{J}^\nabla$. On the other hand, $P(\mathcal{J}^{\nabla^+}X) = \Phi^+P(X)$ so, recalling the computation from section 2.1 which showed $[\mathfrak{n}_{_J}^+, \mathfrak{n}_{_J}^+] = 0$, we may conclude $R^D$ is type (1,1). The result now follows by the theorem of Koszul-Malgrange in section 1.3. ■

Notice we do not say those vector bundles are holomorphic subvector bundles of $T\mathcal{J}(M, \omega)$.

Finally, after assigning a metric to any real surface, all previous constructions follow and we are left with a new tool in the theory of Riemann surfaces: letting $\mathcal{F}$ denote one of the sheaves of germs of holomorphic sections of $\mathcal{H}^\nabla$ or $\mathcal{V}$, then

$$R^1\pi_*\mathcal{F}$$

may tell us something new about $M$.

## 3.5 A simple generalisation

Let us see a generalisation of those results introduced in [20], which were presented in section 2.2 and had in view the real differential geometry of twistor spaces.

Assume $V_0$ is a vector space, $G \subset GL(V_0)$ is a Lie group and $\mathfrak{g}$ is its Lie algebra. Let $Y \subset \mathrm{End}\, V_0$ be a symmetric $G$-space and assume $Y$ is realised as

$$\frac{G}{H} = G \cdot y_0$$

for some fixed element $y_0$, where $\cdot$ denotes the adjoint action. Let $\mathfrak{h}$ denote the Lie algebra of the stabiliser subgroup $H$ of $y_0$, so that $G \to Y$ is a principal $H$-bundle. We also have that

$$TY = G \times_H \mathfrak{m}$$

where $\mathfrak{m}$ is such that $\mathfrak{m} + \mathfrak{h}$ is a symmetric space decomposition of $\mathfrak{g}$. It is known that $\mathfrak{h} = \ker \operatorname{ad}(y_0)$ and

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \qquad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \tag{3.2}$$

Here we assume the further identity $\mathfrak{m} = [\mathfrak{g}, y_0]$.

**Remark.** The last assumption is always satisfied if $G$ is compact (*cf.* [22]). Indeed, with a $G$-invariant metric, we have $\ker \operatorname{ad}(y_0) \perp \operatorname{im} \operatorname{ad}(y_0)$. But the case of $J(V, \omega, 0)$ proves the condition to represent a more general situation. Notice for future reference that, also from [22], we learn that if $G$ is compact then $Y$ has a complex $G$-structure: again, $\operatorname{ad}(y_0)$ is skew symmetric, so $\mathfrak{m}^+$ is the sum of the eigenspaces whose eigenvalue is above the real axis.

Let $F$ be a principal $G$-bundle over a real manifold $M$. Then we have a commutative diagram of fibre bundles

$$\begin{array}{ccc} F & \xrightarrow{\pi_1} & Z = F \times_G Y \\ \pi_0 \searrow & & \swarrow \pi \\ & M & \end{array}$$

where two of them are principal and where $\pi_1(p) = p \cdot y_0$.

As we mentioned in the beginning of section 1.2, if $\alpha$ is a connection on $F$, then there is a horizontal distribution tangent to $Z$: it is defined as $\mathcal{H}^\nabla = \pi_{1*} \ker \alpha$. By general principles, it is smooth and hence it underlies some vector bundle. Now, let $\alpha$ be given and let $\nabla$ denote the associated covariant derivative on $V = F \times_G V_0 \to M$.

**Proposition 3.6.** *The structure group of the vector bundle $\pi^* V \to Z$ reduces to $F \xrightarrow{\pi_1} Z$, so we may write $\pi^* V = F \times_H V_0$ with $V_0$ as a $H$-module. Hence,* $\operatorname{End} \pi^* V$ *admits a global section defined by $\Phi_{p \cdot y_0} = p \cdot y_0$, which is said canonical. Moreover,*

$$\mathcal{H}^\nabla = \left\{ X \in TZ : \ \pi^* \nabla_X \Phi = 0 \right\}.$$

*Proof.* We start by showing that $(F,\pi_1,Z)$ is a reduction of $\pi^*F\to Z$. We have

$$\begin{array}{ccl} F & \xrightarrow{r} & \pi^*F=\big\{(z,p):\ \pi(z)=\pi_0(p)\big\} \\ \downarrow & \swarrow & \\ Z & & \end{array}$$

where $r=(\pi_1,\mathrm{Id})$ is an embedding and $r(pg)=r(p)g=(z,pg)$, for all $g\in H$. Hence the first part.

The connection $\pi^*\nabla$ also reduces to $F$ — though, here, as a $H$-connection with 1-form the very same $\alpha=(\mathrm{pr}_2\circ r)^*\alpha$. Now, let $s$ denote any section of $P$ over some open subset of $Z$. The duality $\pi_1\circ s(z)=z=s(z)\cdot y_0$ yields

$$\pi^*\nabla_X\Phi \;=\; (\pi^*\nabla)_X s\cdot y_0 \;=\; s\cdot[s^*\alpha(X),y_0]$$

by formula (1.5) and the starting observations in section 1.2. Hence

$$\begin{aligned} X\in\ker\,\pi^*\nabla_{\cdot}\Phi \quad&\Longleftrightarrow\quad [\alpha(s_*X),y_0]=0 \\ &\Longleftrightarrow\quad \alpha(s_*X)\in\mathfrak{h} \\ &\Longleftrightarrow\quad s_*X\in\ker\,\alpha+\ker\,\pi_{1*} \\ &\Longleftrightarrow\quad X=\pi_{1*}s_*X\;\in\;\mathcal{H}^\nabla=\pi_{1*}\ker\alpha \end{aligned}$$

(notice $\ker\alpha$ is the kernel corresponding to $\nabla$). ∎

The proof clearly follows that of proposition 2.5. Moreover,

$$TZ=\mathcal{H}^\nabla\oplus\mathcal{V} \tag{3.3}$$

where $\mathcal{V}=\ker\mathrm{d}\pi$, and with both vector bundles sitting in

$$0\longrightarrow\mathcal{V}\longrightarrow TZ\xrightarrow{\mathrm{d}\pi}\pi^*TM\longrightarrow 0.$$

Furthermore, we can also see that $\mathcal{V}=F\times_H\mathfrak{m}$, so it identifies with a subvector bundle of $\mathrm{End}\,\pi^*V$.

Now assume $Y$ to be a complex $G$-symmetric space (*cf.* with the remark above), so that such structure is carried over to the bundle's fibres, and let $M$ have an integrable complex structure $J$. Then we can use the isomorphism $\mathrm{d}\pi:\mathcal{H}^\nabla\to\pi^*TM$ and transport $\pi^*J$ to a complex structure $\mathcal{J}^h$ on the horizontal distribution.

Like in twistor theory, the kählerian complex structure of $Y$ is carried fiberwise to a smooth endomorphism of $\mathcal{V}$ with square $-1$, which we denote by $\mathcal{J}^v$. Finally, we may define an almost-complex structure over the manifold $Z$:

$$\mathcal{J}_3^\nabla = (\mathcal{J}^h, \mathcal{J}^v)$$

preserving the splitting (3.3). In this case, $\pi$ is always pseudo-holomorphic; there is no 'twisting' here.

Just like in twistor theory and because $\mathfrak{m}$ admits a fixed complex structure, we deduce the following result. Let $\rho_{\mathfrak{m}^+}$ denote the component in $\mathfrak{m}^+$ of the curvature 2-form $\rho$ over the principal bundle $F$.

**Theorem 3.9.** *The integrability equation of $\mathcal{J}_3^\nabla$ is equivalent to*

$$\rho_{\mathfrak{m}^+}(u, v) = 0$$

*for all $u, v \in TF \otimes \mathbb{C}$ whose image under $\mathrm{d}\pi_0$ is in $T^-M$.*

*Proof.* The pull-backs to $F$ of (1,0)-forms on $Z$ are spanned by *the* components of $\mathrm{d}\pi_0$, that is up to a chart of $M$, and the components of $\alpha_{\mathfrak{m}^+}$. Then

$$\begin{aligned} \mathrm{d}\alpha_{\mathfrak{m}^+} \;=\; (\mathrm{d}\alpha)_{\mathfrak{m}^+} \;&=\; \rho_{\mathfrak{m}^+} - (\alpha\wedge\alpha)_{\mathfrak{m}^+} \\ &=\; \rho_{\mathfrak{m}^+} \;+\; (2,0),(1,1)-\text{forms}, \end{aligned}$$

as the equations (3.2), $[\mathfrak{m}^+, \mathfrak{m}^+] \subset \mathfrak{m}^+$ and the decomposition $\alpha = \alpha_{\mathfrak{m}^+} + \alpha_{\mathfrak{m}^-} + \alpha_{\mathfrak{h}}$ will easily show. Finally, recalling that $\mathcal{J}_3^\nabla$ is integrable iff the exterior derivatives of (1,0)-forms do not have (0,2)-component and recalling $\rho$ vanishes on vertical tangent vectors, we may conclude. ∎

**Example.** Let $M$ be a complex manifold and $V \to M$ any (oriented) vector bundle of rank $2k$. Suppose $V$ is endowed with a metric $g$ and a respective metric connection $\nabla$. Then we may consider the bundle

$$\mathcal{J}_{(+)}(V, g),$$

of all the $g$-compatible, (orientation preserving) complex structures of $V$, together with its a.c. structure $\mathcal{J}_3^\nabla$. The integrability condition is thus given by the new equation

$$I^+R(J^-X, J^-Y)I^- = 0,$$

for all $X, Y \in TM$ and all $I \in \mathcal{J}_{(+)}(V, g)$. Notice that, if we assume $V \to M$ is a holomorphic vector bundle, then the embedding $\mathcal{J}_{(+)}(V, g) \subset \operatorname{End} V$ is certainly not holomorphic.

Continuing our generalisation of results from [20] or section 2.2, and following formula (3.3), we again assume the conjugacy class $Y$ is just a real symmetric space and $\mathfrak{m} = [\mathfrak{g}, y_0]$. Now we let $P : TZ \to \mathcal{V}$ denote the projection map with kernel the horizontal distribution. $P \in \mathrm{A}^1(\mathcal{V})$, so it may be used to translate $\pi^*\nabla$ to a new connection

$$D^1 = \pi^*\nabla - P,$$

verifying $D^1\Phi = \pi^*\nabla\Phi - [P, \Phi] = 0$ and thus

$$\begin{aligned} D^1_X\mathcal{V} &= D^1_X[\Phi, \operatorname{End}\pi^*V] \\ &= [\Phi, D^1_X\operatorname{End}\pi^*V] \subset \mathcal{V}. \end{aligned}$$

Now assume we are given a linear connection $\nabla^M$, that is, on the tangent bundle of $M$. Using the isomorphism $\mathcal{H}^\nabla \xrightarrow{\mathrm{d}\pi} \pi^*TM$ we may transport $\pi^*\nabla^M$ to another connection $D^2$ on $\mathcal{H}^\nabla$. Then

$$D = D^2 \oplus D^1$$

is a linear connection over $Z$, preserving the splitting (3.3). Notice here a divergence from the theory in section 2.2: there is no $-P$ in the horizontal side.

**Proposition 3.7.** *The connection $D$ has torsion whose vertical part is the projection of $\pi^*R^\nabla$ into $\mathcal{V}$, and whose horizontal part* is $\pi^*T^{\nabla^M}$.

*Proof.* This is exactly as in the proof of theorem 3.2. For the vertical part, though in a general setting, the same arguments carry through. For the horizontal, and since we did not show the second part of that theorem's proof, which is found in [20], we do the proof here up to the remarked minor difference. Let $X, Y \in TZ$. Then

$$\begin{aligned} \mathrm{d}\pi(T^D(X,Y)) &= \pi^*\nabla^M_X \,\mathrm{d}\pi Y - \pi^*\nabla^M_Y \,\mathrm{d}\pi X - \mathrm{d}\pi[X,Y] \\ &= \pi^* T^{\nabla^M}(\mathrm{d}\pi X, \mathrm{d}\pi Y) \end{aligned}$$

since we notice this is a tensor. ∎

Using Frobenius' theorem, the following is now easy to prove.

**Proposition 3.8.** *The distribution $\mathcal{H}^\nabla$ is integrable if and only if $[\pi^* R^\nabla, \Phi] = 0$.*

We remark that, as we saw in the twistor theory case, a section $\psi : M \to Z$ is parallel if and only if $\psi_* TM \subset \mathcal{H}^\nabla$. Indeed, this is now trivial to check:

$$\begin{aligned} \psi^*(\pi^*\nabla_{\psi_* X}\Phi) &= (j^*\pi^*\nabla)_X \;\psi^*\Phi \\ &= \nabla_X \psi \end{aligned}$$

since $\pi \circ \psi = \mathrm{Id}$ and $\psi^*\Phi = \psi$.

Finally we present the example of a symplectic connection which is not locally Kähler — a notion introduced just before proposition 3.1. This can be put in contact with the above by taking the case of the twistor space $\mathcal{J}(M, \omega, *)$ and proving that the horizontal distribution does not admit one single leaf.

In the real functions $a, b, c, d \in \mathrm{C}^\infty_{\mathbb{R}^2}$ of section 1.4, which determine uniquely any symplectic connection on $\mathbb{R}^2$, we choose $a = b = d = 0$ and $c = 1$. So

$$\nabla_{\partial_x}\partial_x = 0, \qquad \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = \partial_x, \qquad \nabla_{\partial_y}\partial_y = -\partial_y$$

Then

$$R(\partial_x, \partial_y)\partial_x = \left(\nabla_{\partial_x}\nabla_{\partial_y} - \nabla_{\partial_y}\nabla_{\partial_x}\right)\partial_x = 0$$

and

$$R(\partial_x, \partial_y)\partial_y = \left(\nabla_{\partial_x}\nabla_{\partial_y} - \nabla_{\partial_y}\nabla_{\partial_x}\right)\partial_y = -2\partial_x.$$

$R_{\partial_x,\partial_y}$ is a nilpotent endomorphism, so it cannot be taking values on $\mathfrak{so}(2)$ as it should if the connection $\nabla$ were reducible.

# Appendix A

Here is the proof of proposition 2.2, section 2.1, which does not recur to $V^c$. Recall also the remark on the sign rule just before that proposition.

**Proof.** *(i)* Everything amounts to show that it is possible to find a basis both symplectic and $g_J$-orthogonal. Recall that a basis $\{X_m\}$ is called symplectic if the matrix of $\omega$ is

$$J_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

so that $X_{m+n} = J_0 X_m$ for $m \leq n$ and $J_0$ *is* a complex structure.

Let $X, Y \in V \backslash \{0\}$ be such that $\omega(X,Y) = \pm 1$ (there exists such a pair). Now, with the restriction of the non-degenerate $\omega$,

$$V_1 = \big\{U : \ \omega(X,U) = \omega(Y,U) = 0\big\} = \{X,Y\}^{\omega}$$

is a symplectic vector space. Admitting by induction the result true for $n-1$, we find the new basis on $V_1$ and a complex structure on $V_1$. Rearranging terms together with $X = -JY$ and $Y = JX$ we find the full basis we required, the index remaining a combinatorial problem. Also $V_1$ becomes the orthogonal complement of $\{X,Y\}$. In the meanwhile, we proved the existence of symplectic bases and the existence of compatible complex structures for any $l$.

*(ii)* Now let $J$ be given and let

$$Q_l = \begin{bmatrix} 1_{n-l,l} & \\ & 1_{n-l,l} \end{bmatrix}$$

where $1_{n-l,l}$ is the matrix of the inner product $x_1y_1 + \ldots + x_{n-l}y_{n-l} - x_{n-l+1}y_{n-l+1} - \ldots - x_ny_n$. So far we have proved there exists $g \in Sp(V,\omega)$ which transforms, say, the first symplectic basis $\{X_m\}$ to a new one both orthogonal and symplectic. Henceforth $g$ satisfies the equations

$$g^t J_0 g = J_0 \qquad \text{and} \qquad -g^t J_0 J g = Q_l.$$

Thus $g^t J_0 = J_0 g^{-1}$, and hence $g^{-1} J g = J_0 Q_l$. It is trivial to see $J_0$ commutes with $Q_l$, so we got a complex structure $J_0 Q_l$ in the same $J(V,\omega,l)$ as $J$. ■

Notice that the proof above to find symplectic bases works just fine in order to find the same in a completely $\mathbb{C}$-framework.

In the following we shall take care of the statements at the end of section 2.1 which were left to prove. To simplify notation let $V = \mathbb{R}^{2n}$ so that $Sp(V^c,\omega) = Sp(2n,\mathbb{C}) = G^C$. Now, $G^C$ acts holomorphically on the grassmannians $Gr(k,2n)$ of complex $k$-planes (not transitively). Notice that the stabilizer $P_\Pi$ of a point $\Pi$ is clearly a complex subgroup.

We are particularly interested in the case $k = n$. A quick computation shows that this $P$ is neither the smallest nor the biggest subgroup we can achieve with those actions (is the smallest subgroup the case when $k = [n/2]$ odd?). However, our $P$ is still big enough for the maximal solvable subalgebra $\mathfrak{r}$ in the Lie algebra $\mathfrak{p}$ of $P$ to be maximal solvable in $\mathfrak{sp}(2n,\mathbb{C}) = \mathfrak{g}^C$.

**Proof.** Just from the theory of solvable Lie algebras on algebraically closed fields, which we are not able to recall here exhaustively, one concludes all solvable $\mathfrak{s}$ preserve some $n$-plane (thinking of $\mathfrak{s} \subset \mathfrak{gl}(\mathbb{C}^{2n})$, its elements are all representable in triangular form for a same basis). So any maximal $\mathfrak{s} \subset \mathfrak{g}^C$ will preserve some $n$-plane. By conjugation with some $g \in GL(2n,\mathbb{C})$ we find $\mathfrak{s}^g \subset \mathfrak{p}$ and $\mathfrak{s}^g = \mathfrak{r}$ to be maximal in $\mathfrak{g}^C$. To see that inclusion, notice the subgroup $P$ coincides with the stabilizer of the $\mathfrak{r}$-stable $n$-plane. ■

$P$ is thus a parabolic subgroup, and has the 'right' dimension. After finding this dimension, the reader may notice that all conclusions we draw next will remain true without referring to flag manifolds' theory. Though, the latter came first to our knowledge than the former.

We need to be more practical to see that dimension. Consider the plane $\Pi = \{(x,0):\ x \in \mathbb{C}^n\}$. Then

$$\begin{aligned} P &= \left\{ \begin{bmatrix} a & ae \\ 0 & {a^{-1}}^t \end{bmatrix} : \ a \in GL(n,\mathbb{C}),\ e = e^t \in \mathfrak{gl}(n,\mathbb{C}) \right\} \\ &= GL(n,\mathbb{C}) \rtimes \mathbb{C}^{\frac{n(n+1)}{2}}. \end{aligned}$$

**Proof.** Let $g \in P$.

$$g \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}$$

implies $c = 0$. Also $g \in G^C$, so

$$\begin{bmatrix} a^t & 0 \\ b^t & d^t \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & -a^t \\ d^t & -b^t \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and we find $d = {a^{-1}}^t$, $d^t b = b^t d$. Equivalently, $d = {a^{-1}}^t$ and $b = ae$ with $e^t = e$. ■

(One may guess that the radical of $P$ is a Borel subalgebra of $\mathfrak{sp}(2n,\mathbb{C})$).

Now we take a digression on the real forms of $G^C$, some of which we do not need. First we see

$$P \cap U(2n-2l, 2l) = U(n-l, l).$$

**Proof.**

$$\begin{bmatrix} a^t & 0 \\ (ae)^t & a^{-1} \end{bmatrix} Q_l \begin{bmatrix} \overline{a} & \overline{ae} \\ 0 & \overline{{a^{-1}}^t} \end{bmatrix} = \begin{bmatrix} a^t 1_{n-l,l}\overline{a} & a^t 1_{n-l,l}\overline{ae} \\ (ae)^t 1_{n-l,l}\overline{a} & * \end{bmatrix} = Q_l$$

iff $a \in U(n-l,l),\ e = 0$. ■

Since $G^C/P$ is connected, arguing with dimensions we find

$$\frac{G^C \cap U(2n-2l, 2l)}{U(n-l,l)} \subset \frac{G^C}{P} = \frac{Sp(n)}{U(n)},$$

where $G^C \cap U(2n-2l, 2l)$ are non-compact real forms of $G^C$ and $Sp(n) = G^C \cap U(2n) = U(n, \mathbb{H})$ is the well known compact real form of $G^C$ (referred in the text). The last equality holds from the fact that the orbit of $Sp(n)$ in $G^C/P$ must be open and closed.

On the other hand, orbits of $Sp(2n, \mathbb{R})$ are only locally closed. The open ones, the flag domains, certainly appear when $H = P_\Pi \cap Sp(2n, \mathbb{R})$ has the lowest possible dimension as $\Pi$ varies. From the above we know this has to be $n^2$. Depending on the signature over $\Pi$ of $h(w_1, w_2) = i\omega(w_1, \overline{w}_2)$, leads to the solutions $H_l = U(n-l, l)$. The $n$-planes are spanned by $\{e_1 + if_1, \ldots, e_{n-l} + if_{n-l}, e_{n-l+1} - if_{n-l+1}, \ldots, e_n - if_n\}$ where $\{e_i, f_j\}$ is a symplectic basis.

Thus we may claim to have constructed a *holomorphic* embedding

$$J(V, \omega, l) = \frac{Sp(2n, \mathbb{R})}{U(n-l, l)} \longrightarrow \frac{G^C}{P}$$

and the commutative diagram in page 47: notice the first $P$ was the stabilizer of a lagrangian plane, since $(x, 0)J_0(y, 0)^t = (x, 0)(0, y)^t = 0$.

Now we proceed with the proof of another statement in section 2.2: if the almost complex structure $\mathcal{J}^\nabla$ is integrable over some $\mathcal{J}(M, \omega, l)$ then the whole $\mathcal{J}(M, \omega, *)$ is a complex manifold.

**Proof.** We want to show the conditions in theorem 2.2 are valid for all $J \in \mathcal{J}(M, \omega, *)$ if they are valid *over some* $\mathcal{J}(M, \omega, l)$. Let $x \in M$ and $V = T_xM$. Consider first the torsion condition:

$$J^+T(J^-X, J^-Y) = 0 \tag{4}$$

$\forall X, Y$ and for all $J \in J(V, \omega, l)$. Fix $J_0$ in this set. Then (4) is saying that $T$ is taking values in the largest $G = Sp(V, \mathbb{R})$-invariant subspace of torsion-like tensors such that

$$J_0^+T(J_0^-X, J_0^-Y) = 0.$$

Indeed, recalling $(g\cdot J_0)^+ = g\cdot J_0^+$ from theorem 2.4, we have that

$$\begin{aligned} J_0^+(g^{-1}\cdot T)(J_0^-X, J_0^-Y) &= J_0^+g^{-1}T(gJ_0^-X, gJ_0^-Y) \\ &= g^{-1}(g\cdot J_0)^+T((g\cdot J_0)^-gX, (g\cdot J_0)^-gY) \end{aligned}$$

and hence $T$ is in the subspace iff $T$ satisfies (4). These are the ideas of [20].

Notice the subspace is in 1-1 correspondence with a $G$-space $\mathcal{T}$ of complex linear tensors defined by the same condition; such mapping is induced by

$$\wedge^2 V\otimes V \longrightarrow \wedge^2 V^c\otimes_c V^c.$$

Now, since we can pass to another $J(V,\omega,l')$ by acting on $J_0$ with an element of $G^C - G$, we just have to prove $\mathcal{T}$ is also $G^C$-invariant. Notice furthermore that analogous arguments follow for the curvature condition, this time with the $G$-subspace siting in

$$\wedge^2 V^c\otimes_c S^2V^c$$

because $\mathfrak{sp}(V,\mathbb{R}) = S^2V$.

Finally, the theory of representations says the irreducible $G$-subspaces are again $\mathbb{C}$-isomorphic to some

$$\wedge^q V^c\otimes_c S^pV^c,$$

hence also $G^C$-invariant. So we find that both $\mathcal{T}$ and the curvature induced subspace must satisfy the required condition of $G^C$-invariance. ■

In conclusion, representation theory is very, very powerful! And there was never great advantage for symplectic geometry in considering the whole $\mathcal{J}(M,\omega,*)$, more than its 0-component.

# Bibliography

[1] A. Andreotti and H. Grauert. Théorème de finitude pour la cohomologie des espaces complexes. *Bull. Soc. Math. France*, 90:193–259, 1962.

[2] M. Atiyah. *Geometry of Yang-Mills fields.* Scuola Normale Superiore, Pisa, 1979.

[3] M. Atiyah, N. Hitchin, and I. Singer. Self-duality in four-dimensional riemannian geometry. *Proc. Roy. Soc. London*, A 362(1711):425–461, 1978.

[4] M. Audin and J. Lafontaine, editors. *Holomorphic curves in symplectic geometry.* Progress in Mathematics, 117. Birkhäuser Verlag, Basel, 1994.

[5] R. Bott and L. Tu. *Differential forms in algebraic topology.* Graduate Texts in Mathematics, 82. Springer-Verlag, New York-Berlin, 1982.

[6] F. Bourgeois and M. Cahen. A variational principle for symplectic connections. *J. Geom. Phys.*, 3:233–265, 2000.

[7] F. Burstall and J. Rawnsley. *Twistor theory for riemannian symmetric spaces.* Lec. Notes in Math., 1424. Springer, Berlin, 1990.

[8] M. Cahen, S. Gutt, J. Horowitz, and J. Rawnsley. Homogeneous symplectic manifolds with ricci-type curvature. *J. Geom. Phys.*, 2:140–151, 2001.

[9] M. Cahen, S. Gutt, J. Horowitz, and J. Rawnsley. Moduli space of symplectic connections of ricci type on $t^{2n}$; a formal approach. Warwick Preprint, February, 2002.

[10] M. Cahen, S. Gutt, and J. Rawnsley. *Symmetric symplectic spaces with Ricci-type curvature*. Kluwer Acad., Math. Phys. Stud., 22, 2000. Conférence Moshé Flato 1999, Vol. II (Dijon).

[11] S. Donaldson and P. Kronheimer. *The geometry of four-manifolds*. Oxford Mathematical Monographs, Oxford University Press, New York, 1990.

[12] A. Douady and J.-L. Verdier, editors. *Les équations de Yang-Mills*, Astérisque, 71–72, Séminaire E. N. S., 1977-1978. Soc. Math. de France, Paris, 1980.

[13] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*. Academic Press, Inc., 1978.

[14] F. Hirzebruch. *Topological methods in algebraic geometry*. Springer-Verlag, Berlin, 1995.

[15] N. J. Hitchin. Kählerian twistor spaces. *Proc. London Math. Soc. (3)*, 43(1):133–150, 1981.

[16] S. Kobayashi. *Differential geometry of complex vector bundles*. Iwanami Shoten, Princeton University Press, 1987.

[17] S. Kobayashi and K. Nomizu. *Foundations of differential geometry*, volume I and II of *Wiley Classics Library*. John Wiley & Sons, Inc., New York, 1963 and 1969.

[18] D. McDuff and D. Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. Oxford University Press, New York, 1998.

[19] R. Narasimhan. *Analysis on real and complex manifolds*. North-Holland Publishing Co., Amsterdam, 1985.

[20] N. O'Brian and J. Rawnsley. Twistor spaces. *Ann. of Global Analy. and Geometry*, 3(1):29–58, 1985.

[21] J. Rawnsley. On the atiyah-hitchin-drinfel'd-manin vanishing theorem for cohomology groups of instanton bundles. *Math. Ann. 241*, 241(1):43–56, 1979.

[22] J. Rawnsley. $f$-structures, $f$-twistor spaces and harmonic maps. *Springer, Berlin*, pages 85–159, 1985. Geometry seminar "Luigi Bianchi" II.

[23] S. Salamon. Harmonic and holomorphic maps. *Springer, Berlin*, pages 161–224, 1985. Geometry seminar "Luigi Bianchi" II.

[24] C. L. Siegel. Symplectic geometry. *Amer. J. Math.*, 65:1–86, 1943.

[25] I. Vaisman. Symplectic curvature tensors. *Monatsh. Math.*, 100(4):299–327, 1985.

[26] I. Vaisman. Symplectic twistor spaces. *J. Geom. Phys.*, 3(4):507–524, 1986.

[27] H. Wu. An elementary method in the study of nonnegative curvature. *Acta Math.*, 142(1-2):57–78, 1979.

At the beginning, 'a Thought to the world' is from Charles Dickens book "Hard Times", *c.* 1850. It was copied from the edition of Penguin Popular Classics, page 146.