Sustainable Growth Under Intertemporally Dependent Preferences *

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Abstract

The formal literature on sustainability tends to adopt a very long run perspective. One assumption often used is that households' preferences are intertemporally independent. This means that consumer demand may (optimally) shift substantially for a change in preferences, technology, or policy incentives. Sustainability has always been studied in this intertemporal independence framework. But if preferences are intertemporal dependent (Ryder and Heal-1973) the resulting cyclical behaviour of consumption along optimal path might jeopardize the altruistic dimension of sustainability, at least during some moments in time.

The process of intertemporal dependent preferences and its influence over consumer behaviour has not yet been fully studied in Environmental Economics. This paper will consider two extensions of the existing models on Growth and Environment by assuming that preferences are intertemporally dependent not only as regard the consumption of goods but also as regards the flow of services provided by stock of natural capital. The results show that we may have a rich array of local dynamics, and, in particular oscillatory transitions. These results depend on both sources of habit formation

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1 Introduction

The formal literature on sustainability tends to adopt a very long run perspective. One of the assumptions that is consistent with that perspective is the assumption that households' preferences are additively separable, or intertemporally independent. This implies that consumer demand may (optimally) shift substantially for a change in preferences, technology, or policy incentives.

However, given the civilizational and cultural contour of preferences towards the environment we may expect a protracted short to medium run adjustment of consumption and of related conservationist policies.

There are several ways to build intertemporal dependent preferences. In the recent literature on macroeconomics, both closed and open, and finance the Ryder and Heal (1973) model for the so-called habit formation has been used to solve a number of puzzles. And it conforms well with the short run low volatility of consumption. Intuitively, it says that the consumer alongside with current consumer builds a stock of habits. This stock of habits is no more than a weighted sum of the history of past consumption. This introduces some degree of inertia in current consumption.

The literature has also highlighted the differences in the dynamics of the adjustment depending upon the intertemporal complementarities in the consumption at different periods: adjacent or distant complementarity. The first case has been used as a model for additive or compulsive behaviors.

In a setup in which preferences depend both on the consumption of goods (that use a certain amount of natural resources in their production) and on the level of the natural capital, the introduction of intertemporally dependent preferences departs slightly from the literature, which tends to consider just one good.

There are two dimensions involved. The consumer may have intertemporally interdependent preferences as regards the consumption of the good, as regards the utility of the services stemming from the stock of natural capital, or both. The first will mean that households take account of habits in the consumption of material goods and the second that they do it in their perception of the utility derived from the natural environment.

In this paper we will use the most flexible structure in order to capture possibly different intertemporal dependencies as regards consumption and "greenness". We use a centralized economy setup. We abstract from other capital inputs, both physical and human.

2 The model

Our model features unbounded growth, driven by exogenous productivity growth, in an economy that uses a bounded stock of natural capital. We use the simplest technology and the simplest dynamics for the natural capital, in order to highlight the structure of the preferences.

The preferences are assumed to have three main dimensions. First, preferences depend on both the services extracted from the consumption of manufactures goods and from the environment. This assumption is common in the literature. Second, preferences are intertemporally dependent, that is, the marginal rate of substitution for the services extracted from consumption between two different moments is not independent from the path of consumption. This means that the change in consumption has some may have some inertia. We use that habit formation model in the tradition of Ryder and Heal (1973). Third, the kind of intertemporal dependence related to the consumption of manufacturing goods may be different from that of the consumption of environmental services.

We assume that while the consumption of manufacturing goods displays adjacent complementarity, the consumption of environmental services displays distant complementarity. This means that while consumers tend to be addictive as regards the consumption of goods, they are willing to trade off reductions in the immediate consumptions of environmental services against an increase in the near future.

Next we will present the model starting with the technology part and dealing next with the preferences.

2.1 Technology and the environment

We assume an economy in which production uses only (the flow of) natural capital, $P := \{P(t), 0 \leq t < \infty\}$, according to the production function

$$Y(t) = A(t)P(t)$$

where $A := \{A(t), 0 \leq t < \infty\}$ is a growing factor. It may be related to the growth of population or of human capital or both or with productivity. The important thing to note is that we assume that A grows exponentially at a constant rate γ. Formally $\dot{A}(t) = \gamma A(t)$. Even though we may recast our model in an endogenous growth framework, we assume that the forces that may generate unbounded growth are exogenous.

The use in production, diminishes the stock of natural capital, N. The simplest dynamics of the natural capital is given by the equation (see Smulders (1999))

$$\dot{N}(t) = E(N(t)) - P(t) \tag{1}$$

where $E(N)$ represents the reproduction of the natural capital.

A possible specification is

$$E(N(t)) = N(t)^{\phi} - \mu N(t)$$

where $0 \leq \phi \leq 1$, and $0 \leq \mu \leq 1$ is the natural depreciation rate.

As is common in the natural resources literature (see ...) we assume that natural capital is bounded. Therefore, it will grow asymptotically at a zero growth rate. Therefore $\gamma_N = \gamma_P = 0$.

Then the asymptotic growth rate of production is $\gamma_y = \gamma_A = \gamma$. If we represent the detrended level of production by a small letter, then the level of production, at time t, is

$$Y(t) = y(t)e^{\gamma_y t} = a(t)P(t)e^{\gamma t},$$

where $a(t) = A(0)$, for all $t \geq 0$.

The production in this economy is only used in consumption. The equilibrium in the goods market then becomes

$$Y(t) = C(t). \tag{2}$$

Accordingly, the level of consumption should be equal to $C(t) = c(t)e^{\gamma t}$, meaning that consumption will fluctuate in the short run around a long run trajectory that will grow unboundedly at a rate γ.

2.2 Preferences

The representative consumer determines the optimal path of consumption in order to maximize an intertemporal utility function. However, as we are particularly concerned with the dynamics of this economy, we will assume an intertemporally dependent preference structure, or habit formation utility function. We assume that the representative consumer attaches value to both the consumer of material goods and to the stock of natural resources.

Let the paths of consumption of manufactured goods and of the natural capital be denoted by $C := \{C(t) : 0 \leq t < \infty\}$, and $N := \{N(t) : 0 \leq t < \infty\}$. The intertemporal utility function is the functional over the instantaneous flows of utility, discounted at the positive and constant rate δ,

$$V(C, N) := \int_0^{+\infty} u(C(t), S(t), \varphi(N(t)), Z(t))e^{-\delta t}dt$$

where

$$S(t) = S(0)e^{-\rho_1 t} + \rho_1 \int_0^t e^{-\rho_1 \tau} C(\tau)d\tau \tag{3}$$

$$Z(t) = Z(0)e^{-\rho_2 t} + \rho_2 \int_0^t e^{-\rho_2 \tau} \varphi(N(\tau))d\tau, \tag{4}$$

are the stock of habits related to the consumption of manufactured goods and to the amenity services extracted from the natural capital. We assume that the instantaneous flow of services extracted from the stock of natural capital is a increasing function of the stock of natural capital, $\varphi(.)$. The coefficients ρ_1 and ρ_2 measure both the rate of decay of the habits related to the consumption of goods and the amenity services and the rate of habit formation from the current flow of services consumed.

Equations (3) and (4) may be written in differential form as

$$\dot{S}(t) = \rho_1(C(t) - S(t)) \tag{5}$$
$$\dot{Z}(t) = \rho_2(\varphi(N(t)) - Z(t)). \tag{6}$$

These equations feature the variables in levels. As C and N grow asymptotically at the rates γ and zero respectively, then the asymptotic rate of growth should be equal to γ for S and zero for Z. Then $S(t) = s(t)e^{\gamma t}$.

In addition to the regularity, intratemporal and intertemporal requirements for the utility function, the need to separate trend and cycle, impose additional properties on the utility function:

First, we assume that the utility function is continuous in all its arguments.

Second, the services extracted from natural capital are of a particular nature, that we cannot amalgamate with the services extracted from produced goods. Therefore, differently from the (few) previous cases of habit formation models with two goods in the literature [1], we assume that the utility function is weakly separable between produced goods and natural services, as $u(u_1(C, S), u_2(N, Z))$ [2]. We depart from the previous literature in habit formation, that separates services from habits, as in $u(g_1(C, N), g_2(S, Z))$. We do not consider an aggregate index of total consumption, and consider that the stock of habits is represents the past consumption of that aggregate, but rather, we define different habit formation processes for consumption and the services of natural capital.

Additionally, we want to explore in this paper the consequences of different habit formation processes for both types of utility.

Third, we assume that $u(.)$ is homogeneous. In order to have unbounded growth in consumption, it is well know in the endogenous growth literature (see Rebelo (1991)) that the utility function should be homogeneous. Therefore, the two sub-utility functions, $u_1(.)$ and $u_2(.)$, should also be homogeneous. However, as their arguments grow asymptotically at different rates, $u(.)$ cannot be additively separable. Unless we use a very particular, and therefore unlikely, structure for the sub-utility functions, the utility function would not be homogeneous in the additive case.

Therefore, the instantaneous utility function has the following representation:

$$u\left(u_1(C(t), S(t)), u_2(N(t), Z(t))\right) = u\left(u_1(c(t), s(t)), u_2(N(t), Z(t))\right) e^{\gamma_u t}$$

where γ_u is the asymptotic rate of growth of utility. Then, intertemporal utility functional is a function of the paths of the detrended variables $c = \{c(t), 0 \leq t < \infty\}$ and N,

$$V(c, N) := \int_0^{+\infty} u(c(t), s(t), N(t), Z(t))e^{-\delta^* t} dt, \tag{7}$$

[1] In particular in international macroeconomics, see Mansoorian (1993).

[2] Weakly separability exists in this case exists if $\frac{\partial}{\partial N}\left(\frac{\partial C}{\partial S}\right) = \frac{\partial}{\partial Z}\left(\frac{\partial C}{\partial S}\right) = \frac{\partial}{\partial C}\left(\frac{\partial N}{\partial Z}\right) = \frac{\partial}{\partial S}\left(\frac{\partial N}{\partial Z}\right) = 0$.

where we assume that the instantaneous utility is actualized by the rate

$$\delta^* := \delta - \gamma_u > 0, \tag{8}$$

and, use the simplified form $\varphi(N) = N$.

The type of weak separability that we have assumed thus far characterizes part of the intratemporal structure of preferences. However, in a model with an habit preference structure the growth and curvature properties of the instantaneous utility function depend on the nature of the intertemporal preferences.

We have to distinguish between instantaneous or intratemporal marginal utility, that is the instantaneous partial derivatives, from the intertemporal marginal utility. While the first are calculated over all the arguments of the utility function, the later are calculated over the path of consumption of the services of each good. We may characterize formally the (Johnson-) intertemporal marginal utilities by the Fréchet derivatives.

Let a given path of consumption **c** be perturbed by an arbitrary function of time $h := \{h(t), 0 \leq t < \infty\}$. Then, if we interpret the change in the value of consumption as a Fréchet differential, we get

$$V_c(c, N)h = \int_0^\infty \left(u_c(t)h(t) + \rho_1 u_s(t) \int_0^t e^{-(\rho_1+\gamma)\tau} h(\tau) d\tau ds \right) e^{-\delta^* t} dt$$

and for a perturbation of N by a path $k := \{k(t), 0 \leq t < \infty\}$,

$$V_N(c, N)k = \int_0^\infty \left(u_N(t)k(t) + \rho_2 u_Z(t) \int_0^t e^{-\rho_2 \tau} k(\tau) d\tau ds \right) e^{-\delta^* t} dt,$$

where $u_j(t) = \frac{\partial}{\partial j} u(c(t), s(t), N(t), Z(t))$, for $j = c, s, N, Z$. If we assume stationary perturbations over the steady state levels of the variables $\overline{c} := \{c(t) = \overline{c}, 0 \leq t < \infty\}$, $\overline{N} := \{N(t) = \overline{N}, 0 \leq t < \infty\}$, then we get the intertemporal marginal utilities for a permanent shift in consumption

$$V_c(\overline{c}, \overline{N}) = \frac{\xi_c}{\delta^*} \tag{9}$$

$$V_N(\overline{c}, \overline{N}) = \frac{\xi_N}{\delta^*} \tag{10}$$

where

$$\xi_c = u_c + \frac{\rho_1}{\delta^* + \rho_1 + \gamma} u_s \tag{11}$$

$$\xi_N = u_N + \frac{\rho_2}{\delta^* + \rho_2} u_Z. \tag{12}$$

and $\overline{u}_j = u_j(\overline{c}, \overline{s}, \overline{N}, \overline{Z})$, for $j = c, s, N, Z$.

In the seminal paper on habit formation Ryder and Heal (1973), the authors use the Volterra functional derivative, which can be seen as a particular case of the Fréchet derivative such that the perturbations are Dirac's delta. It measures the marginal change in the value function for a change in a particular moment, say $t = t_1$. The expressions for the "needle" variations are

$$\begin{aligned} V_c(c,N)\delta(t_1) &= u_c(t_1)e^{-\delta t_1} + \rho_1 \int_{t_1}^{\infty} e^{-(\rho_1+\gamma)(t-t_1)-\delta^* t} u_s(t)dt \\ V_N(c,N)\delta(t_1) &= u_N(t_1)e^{-\delta t_1} + \rho_2 \int_{t_1}^{\infty} e^{-\rho_2(t-t_1)-\delta^* t} u_Z(t)dt. \end{aligned}$$

If we evaluate the former expressions along stationary paths, we get expressions analogous to (9) and (10), in which the discount factor is $e^{-\delta^* t_1}$, instead of the infinite sum of discounted factors $\frac{1}{\delta^*} = \int_0^\infty e^{-\delta^* t}dt$.

Most papers in the literature assume that the instantaneous utility function is increasing in the flow of consumption and decreasing in the stock of habits, meaning that consumers abhor decreases in consumption. Though we accept this as a natural assumption for the consumption of the physical good, we prefer to characterize the services of the natural capital as a beneficial good (as in Becker and Murphy (1988)), which is similar to the consumption of cultural goods.

Then, we assume, formally, that

$$u_c > 0, \; u_s \leq 0, \; u_N > 0, \; u_Z \geq 0 \tag{13}$$

but we also assume that there is no satiation,

$$V_c > 0, \; V_N > 0, \tag{14}$$

or equivalently that $\xi_c > 0$ and $\xi_N > 0$.

Following Ryder and Heal (1973), the intertemporal marginal rates of substitution, measure the change in a variable i in $t = t_1$ for a unit change in the variable j at $t = t_2 > t_1$

$$R_{ij}(c,N)\delta(t_1)\delta(t_2) = \frac{V_i(c,N)\delta(t_1)}{V_j(c,N)\delta(t_2)}, \;\; i,j = c,N. \tag{15}$$

The intertemporal interdependency was measured by Ryder and Heal (1973) by the change in the marginal rate of substitution, v.g., the change in the rates of substitution between times t_2 and t_1 for unit changes in t_3. However, these authors only considered one argument to the utility function. They only determined the own intertemporal dependency, i.e., the change in the intertemporal marginal rate of substitution between moments t_2 and t_1 for a unit change in consumption in t_3. If the change is positive (negative) then it is said that we have distant (adjacent) complementarity. If the change is equal to zero then we have intertemporal independency. In our case, in addition to the own intertemporal substitution we should also have assumptions relating to the *crossed* intertemporal substitution.

Computing all the partial functional derivatives associated to "needle" changes at

$t = t_3$ in equation (15), we get

$$R_{cc,c} = (\rho_1 + \gamma)\Delta_1 \frac{\xi_{cc}}{\xi_c} \tag{16}$$

$$R_{cc,N} = (\rho_1 + \gamma)\Delta_1 \frac{\xi_{cN}}{\xi_c} \tag{17}$$

$$R_{NN,c} = \rho_2\Delta_2 \frac{\xi_{Nc}}{\xi_N} \tag{18}$$

$$R_{NN,N} = \rho_2\Delta_2 \frac{\xi_{NN}}{\xi_N} \tag{19}$$

where

$$\begin{aligned}
\Delta_1 &= \left(e^{-(\delta^*+\rho_1+\gamma)(t_3-t_1)} - e^{-(\delta^*+\rho_1+\gamma)(t_3-t_2)}\right) e^{\delta^*(t_2-t_1)} < 0 \\
\Delta_2 &= \left(e^{-(\delta^*+\rho_2)(t_3-t_1)} - e^{-(\delta^*+\rho_2)(t_3-t_2)}\right) e^{\delta^*(t_2-t_1)} < 0
\end{aligned}$$

because $t_3 > t_2 > t_1$, and

$$\xi_{cc} = u_{sc} + \frac{\rho_1}{\delta^* + 2(\rho_1 + \gamma)} u_{ss} \tag{20}$$

$$\xi_{cN} = u_{sN} + \frac{\rho_2}{\delta^* + \rho_1 + \gamma + \rho_2} u_{sZ} \tag{21}$$

$$\xi_{Nc} = u_{cZ} + \frac{\rho_1}{\delta^* + \rho_1 + \gamma + \rho_2} u_{sZ} \tag{22}$$

$$\xi_{NN} = u_{NZ} + \frac{\rho_2}{\delta^* + 2\rho_2} u_{ZZ}. \tag{23}$$

The other derivatives related to the intertemporal marginal rates of substitution are

$$R_{cN,c} = \frac{\xi_c}{\xi_N}\left(R_{cc,c} - R_{NN,c}\right) \tag{24}$$

$$R_{cN,N} = \frac{\xi_c}{\xi_N}\left(R_{cc,N} - R_{NN,N}\right) \tag{25}$$

$$R_{Nc,c} = -\frac{\xi_N}{\xi_c}\left(R_{cc,c} - R_{NN,c}\right) \tag{26}$$

$$R_{Nc,N} = -\frac{\xi_N}{\xi_c}\left(R_{cc,N} - R_{NN,N}\right). \tag{27}$$

We assume that $\xi_{cc} > 0$, $\xi_{Nc} < 0$, $\xi_{cN} > 0$ and $\xi_{NN} < 0$. This implies that:

- $R_{cc,c} < 0$ and $R_{cc,N} < 0$, that is, there is own and crossed adjacent complementarity in the consumption of the good: an increase in $c(t_3)$ or in $N(t_3)$ shifts consumption from $c(t_1)$ to $c(t_2)$;
- $R_{NN,N} > 0$ and $R_{NN,c} > 0$, that is, there is own and crossed distant complemen-

tarity in the amenity services: an increase in $N(t_3)$ or in $c(t_3)$ shifts consumption of services from $N(t_2)$ to $N(t_1)$;

- $R_{cN,c} < 0$ and $R_{cN,N} < 0$, that is, there is crossed adjacent complementarity in the consumption of the good as regards the environmental services: an increase in $c(t_3)$ or $N(t_3)$ shifts consumption from $c(t_1)$ to $N(t_2)$;
- $R_{Nc,c} > 0$ and $R_{Nc,N} > 0$, that is, there is crossed distant complementarity in the consumption of environmental services as regards the consumption of the good: an increase in $c(t_3)$ or $N(t_3)$ shifts consumption from $c(t_2)$ to $N(t_1)$.

In the literature there are two major specifications for the sub-utility functions, $u_j(.)$, in habit formation models: Constantinides (1990) assumes an additive function as $u_1(.) := C - \gamma s$, and Abel (1990) or Carroll (2000) use a multiplicative function as $u_1(.) := cs^{-\gamma}$.

Only the last type of utility function is consistent with our previous assumptions. In particular, we assume that

$$u(t) = \frac{\left[\left(c(t)s(t)^{\eta_1}\right)^{\alpha}\left(N(t)Z(t)^{\eta_2}\right)^{\beta}\right]^{1-\sigma}}{1-\sigma} \tag{28}$$

where $\sigma > 1$, $\alpha > 0$, $\beta > 0$, $-1 < \eta_1 < 0$ and $\eta_2 > 0$. This implies that

$$\delta^* \equiv \delta - \gamma_u = \delta - \alpha(1-\sigma)(1+\eta_1)\gamma > 0$$

that

$$\xi_c = \frac{\alpha(1-\sigma)\overline{u}}{\overline{c}}\left(1 + \frac{\eta_1(\rho_1+\gamma)}{\delta^*+\rho_1+\gamma}\right) > 0$$

because $(1-\sigma)\overline{u} > 0$ and the second term as the same sign as $\delta + (1+\eta_1)(\alpha\gamma(\sigma-1) + \gamma + \rho_1) > 0$, and

$$\xi_N = \frac{\beta(1-\sigma)\overline{u}}{\overline{N}}\left(1 + \frac{\eta_2\rho_2}{\delta^*+\rho_2}\right) > 0.$$

It is also easy to see that, with the previous assumptions, that the following holds

$$\begin{aligned}
\xi_{cc} &= \frac{\alpha\eta_1(1-\sigma)\overline{u}}{\overline{cs}}\left[\alpha(1-\sigma)\left(1+\frac{\eta_1(\rho_1+\gamma)}{\delta^*+2(\rho_1+\gamma)}\right) - \frac{\rho_1+\gamma}{\delta^*+2(\rho_1+\gamma)}\right] \\
&> 0 \qquad (29)\\
\xi_{cN} &= \frac{\alpha\beta\eta_1(1-\sigma)^2\overline{u}}{\overline{N}\overline{s}}\left(1+\frac{\eta_2\rho_2}{\delta^*+\rho_1+\gamma+\rho_2}\right) > 0 \qquad (30)\\
\xi_{Nc} &= \frac{\alpha\beta\eta_2(1-\sigma)^2\overline{u}}{\overline{c}\overline{N}}\left(1+\frac{\eta_1(\rho_1+\gamma)}{\delta^*+\rho_1+\gamma+\rho_2}\right) < 0 \qquad (31)\\
\xi_{NN} &= \frac{\beta\eta_2(1-\sigma)\overline{u}}{\overline{N}^2}\left[\beta(1-\sigma)\left(1+\frac{\eta_2\rho_2}{\delta^*+2\rho_2}\right) - \frac{\rho_2}{\delta^*+2\rho_2}\right] > 0. \qquad (32)
\end{aligned}$$

The second order Fréchet derivatives measure the change in the first order derivative of the value for any time-dependent shift in the trajectory of a state variable. If we assume a constant "parallel" shift from stationary paths of c and N, we get

$$\begin{aligned}
V_{cc}(\overline{c},\overline{N}) &= \frac{1}{\delta^*}\left\{u_{cc} + \frac{2\rho_1}{\delta^* + \rho_1 + \gamma}\xi_{cc}\right\} && (33)\\
V_{NN}(\overline{c},\overline{N}) &= \frac{1}{\delta^*}\left\{u_{NN} + \frac{2\rho_2}{\delta^* + \rho_2}\xi_{NN}\right\} < 0 && (34)\\
V_{cN}(\overline{c},\overline{N}) &= \frac{1}{\delta^*}\left\{u_{cN} + \frac{\rho_1}{\delta^* + \rho_1 + \gamma}\xi_{cN} + \frac{\rho_2}{\delta^* + \rho_2}\xi_{Nc}\right\} && (35)\\
V_{Nc}(\overline{c},\overline{N}) &= V_{cN}(\overline{c},\overline{N}). && (36)
\end{aligned}$$

Given our previous assumptions, $V_{NN} < 0$. The other signs are ambiguous. From now on, we will assume that $V_{cc} < 0$.

3 The balanced growth path

The *equilibrium* in this economy is characterized by the paths of consumption and of the stock of natural capital, such that $C(t) = c(t)e^{\gamma t}$ is unbounded and $N(t)$ is bounded. As there are no externalities then it is Pareto optimal and, therefore, it is equivalent to the solution of a centralized problem of optimal intertemporal allocation of consumption and use of natural resources.

The equilibrium paths of the detrended consumption $\hat{c}$ and stock of natural resources $\hat{N}$ should maximize the intertemporal utility unction

$$V(c,N) = \int_0^{+\infty} u(c(t), s(t), N(t), Z(t))e^{-\delta^* t}dt \tag{37}$$

subject to

$$\begin{aligned}
\dot{s}(t) &= \rho_1(c(t) - s(t)) - \gamma s(t) && (38)\\
\dot{Z}(t) &= \rho_2(N(t) - Z(t)) && (39)\\
\dot{N}(t) &= E(N(t)) - \frac{c(t)}{A}, && (40)
\end{aligned}$$

given $s(0) = s_0$, $Z(0) = Z_0$ and $N(0) = N_0$.

The first order conditions for an optimum are the following system

$$\begin{aligned}
0 &= \hat{u}_c + \lambda_1\rho_1 - \frac{\lambda_3}{A} && (41)\\
\dot{\lambda}_1 &= (\delta^* + \rho_1 + \gamma)\lambda_1 - \hat{u}_s && (42)\\
\dot{\lambda}_2 &= (\delta^* + \rho_2)\lambda_2 - \hat{u}_Z && (43)\\
\dot{\lambda}_3 &= (\delta^* - E^{'}(N))\lambda_3 - \rho_2\lambda_2 - \hat{u}_N && (44)
\end{aligned}$$

plus equations (38)-(40), evaluated at the optimal values for the consumption, that is $\hat{u}_j = u_j(\hat{c}, s, Z, N)$ for $j = c, s, N, Z$. The following transversality conditions

$$\lim_{t\to\infty} \lambda_1(t)s(t)e^{-\delta^* t} = \lim_{t\to\infty} \lambda_2(t)Z(t)e^{-\delta^* t} = \lim_{t\to\infty} \lambda_3(t)N(t)e^{-\delta^* t} = 0$$

hold as we assume that $\delta^* > 0$.

The equilibrium (exogenous) *balanced growth path* are the paths of consumption and of the stock of natural capital, $\overline{C}$ and $\overline{N}$ and of the associated habitual and co-state variables, $\overline{S}$, $\overline{Z}$, $\overline{\Lambda}_1$, $\overline{\Lambda}_2$ and $\overline{\Lambda}_3$ such that, $\overline{C}(t) = \overline{c}e^{\gamma t}$, $\overline{N}(t) = \overline{N}$, $\overline{S}(t) = \overline{s}e^{\gamma t}$, $\overline{Z}(t) = \overline{Z}$, $\overline{\Lambda}_1 = \overline{\lambda}_1 e^{(\gamma_u - \gamma)t}$, $\overline{\Lambda}_2 = \overline{\lambda}_2 e^{\gamma_u t}$ and $\overline{\Lambda}_3 = \overline{\lambda}_3 e^{\gamma_u t}$, for all $t \in [0, \infty)$, where $\overline{c}$, $\overline{s}$, $\overline{N}$, $\overline{Z}$, $\overline{\lambda}_1$, $\overline{\lambda}_2$ and $\overline{\lambda}_3$ are the equilibrium of the system (41)-(44) and (38)-(40).

The long run values of the state and co-state variables verify the following conditions. First note that as

$$\begin{aligned} \overline{Z} &= \overline{N} && (45) \\ \overline{c} &= AE(\overline{N}), && (46) \end{aligned}$$

then

$$\overline{s} = \frac{\rho_1 \overline{c}}{\rho_1 + \gamma} = \frac{\rho_1 AE(\overline{N})}{\rho_1 + \gamma}. \tag{47}$$

Then, the shadow prices depend monotonously on N,

$$\begin{aligned} \overline{\lambda}_1 &= \frac{u_s(\overline{N})}{\rho^* + \rho_1 + \gamma} && (48) \\ \overline{\lambda}_2 &= \frac{u_Z(\overline{N})}{\rho^* + \rho_1 + \gamma} && (49) \\ \overline{\lambda}_3 &= A\delta^* V_c(\overline{N}). && (50) \end{aligned}$$

At last the steady state values for the natural capital can be determined from the non-linear equation

$$V_N(\overline{N}) = A\left(\rho^* - E'(\overline{N})\right) V_c(\overline{N}). \tag{51}$$

This equation highlights the usefulness of the definition of the intertemporal utility function: the steady state stock o natural capital is the one that equates the marginal utility of the environment with the capitalized value of marginal utility of consumption, where the actualization rate is equal to the difference between the detrended rate of time preference and of the marginal rate of regeneration. Note that we would get a formally analogous long run condition to eqution (51) in a model with no habit formation,

As $V_c(\overline{N}) > 0$ and $V_N(\overline{N}) > 0$ then, a steady state value for N will only exist if $\overline{N} \in \{N : \rho^* - E'(N) > 0\}$. We will assume next that this condition holds. In this case, as we assumed that both the utility function $u(.)$ and the sub-utility functions, $u_1(.)$ and

$u_2(.)$ are homogeneous, then equation (51) is equivalent to

$$\delta^* - E^{'}(N) = \nu \frac{E(N)}{N}, \tag{52}$$

where ν is a constant. As the left hand side of equation (52) is decreasing everywhere in N and the right hand side is increasing everywhere in N, then there a unique equilibrium point. Therefore if an equilibrium point $\overline{N}$ exists, then it is unique, and an unique equilibrium balanced growth path exists and is unique.

In particular, for the utility function (28), we get

$$\nu = \frac{\xi_N}{\xi_c} = \frac{\beta(\delta^* + \rho_1 + \gamma)[\delta^* + \rho_2(1 + \eta_2)]}{\alpha(\delta^* + \rho_2)[\delta^* + (\rho_1 + \gamma)(1 + \eta_1)]} > 0$$

with our previous assumptions.

4 Local dynamics around the BGP

In this section we will study the local dynamics around the balanced growth path.

As the instantaneous utility function is strictly concave as regards c, then we can use the implicit function theorem and solve locally equation (41) for c, to get the optimal instantaneous consumption function as $\hat{c} = \hat{c}(s, N, Z, \lambda_1, \lambda_3)$, where $\hat{c}_s = -\frac{u_{cs}}{u_{cc}}$, $\hat{c}_N = -\frac{u_{cN}}{u_{cc}}$, $\hat{c}_Z = -\frac{u_{cZ}}{u_{cc}}$, $\hat{c}_{\lambda_1} = -\frac{\rho_1}{u_{cc}}$ and $\hat{c}_{\lambda_3} = \frac{1}{Au_{cc}}$.

The Jacobian associated to equations (42)-(44) and (38)-(40) is,

$$J := \begin{pmatrix} \delta I - H_{22}^T & H_{12} \\ H_{21} & H_{22} \end{pmatrix},$$

where the two sub-matrices in the anti-diagonal are symmetric

$$H_{12} = \begin{pmatrix} -(u_{cs}\hat{c}_s + u_{ss}) & -(u_{cs}\hat{c}_Z + u_{sz}) & -(u_{cs}\hat{c}_N + u_{sN}) \\ -(u_{cs}\hat{c}_Z + u_{sz}) & -(u_{cz}\hat{c}_Z + u_{zz}) & -(u_{cz}\hat{c}_N + u_{zN}) \\ -(u_{cs}\hat{c}_N + u_{sN}) & -(u_{cz}\hat{c}_N + u_{zN}) & -\lambda_3 E^{''} - (u_{NN} + u_{cN}\hat{c}_N) \end{pmatrix},$$

$$H_{21} = \begin{pmatrix} \rho_1\hat{c}_{\lambda_1} & 0 & \rho_1\hat{c}_{\lambda_3} \\ 0 & 0 & 0 \\ \hat{c}_{\lambda_1}A & 0 & -\hat{c}_{\lambda_3}A \end{pmatrix},$$

and

$$H_{22} = \begin{pmatrix} \rho_1(\hat{c}_s - 1) - \gamma & \rho_1\hat{c}_Z & \rho_1\hat{c}_N \\ 0 & -\rho_2 & \rho_2 \\ -\hat{c}_s A & -\hat{c}_Z A & E^{'} - \hat{c}_N A \end{pmatrix}.$$

The charateristic polynomial is

$$c(J,\psi)=\sum_{j=0}^{6}(-1)^{j}M_{6-j}\psi^{j}$$

where M_j is the sum of the principal minors of order $j=0,\ldots,6$, of matrix J, where $M_0=1$. In Brito (1999) it is proved that the principal minors of odd order are functions of the polynomials of even order. The following relationships hold: $M_1=\text{Trace}(J)=3\delta^*$, $M_1=-5(\delta^*)^3+2\delta^*M_2$ and $M_5=3(\delta^*)^5-M_2(\delta^*)^3+M_4\delta^*$. It is also proved that the eigenvalues are

$$\psi_j^i=\frac{\delta^*}{2}(1\mp\sqrt{\omega_j}),\ \ i=s,u,\ \ j=1,2,3,$$

where $\omega_1=1-\frac{k_3}{2}+s_1+s_2$ and $\omega_{2,3}=1-\frac{k_3}{2}-\frac{1}{2}(s_1+s_2)\pm\frac{\sqrt{3}}{2}(s_1-s_2)i$, for $i=\sqrt{-1}$ and $s_{1,2}=\left\{-\left(\frac{k_2}{3}\right)^3+\frac{k_1}{2}\frac{k_2}{3}-\frac{k_0}{2}\pm\sqrt{\Delta}\right\}^{\frac{1}{3}}$ where the discriminant is

$$\Delta=k_0\left(\frac{k_2}{3}\right)^3-\frac{1}{3}\left(\frac{k_1}{2}\right)^2\left(\frac{k_2}{3}\right)^2-k_0\frac{k_1}{2}\frac{k_2}{3}+\left(\frac{k_1}{3}\right)^3+\left(\frac{k_0}{2}\right)^2$$

and $k_0=\left(\frac{\delta^*}{2}\right)^{-6}\det J$, $k_1=\left(\frac{\delta^*}{2}\right)^{-4}\kappa_1$ and $k_2=\left(\frac{\delta^*}{2}\right)^{-2}\kappa_2$, where

$$\kappa_1 = 3(\delta^*)^4-M_2(\delta^*)^2+M4, \tag{53}$$
$$\kappa_2 = -3(\delta^*)^2+M_2. \tag{54}$$

It can be proved that the following relationships between the eigenvalues and the last coefficients hold

$$\psi_1^s\psi_1^u+\psi_2^s\psi_2^u+\psi_3^s\psi_3^u = \kappa_2, \tag{55}$$
$$\psi_1^s\psi_1^u\psi_2^s\psi_2^u+\psi_1^s\psi_1^u\psi_3^s\psi_3^u+\psi_2^s\psi_2^u\psi_3^s\psi_3^u = \kappa_1, \tag{56}$$
$$\psi_1^s\psi_1^u\psi_2^s\psi_2^u\psi_3^s\psi_3^u = \det(J). \tag{57}$$

It is also proved in Brito (1999) that the maximum dimension of the stable manifold is three. Then ψ_i^u, for $i=1,2,3$, have positive real parts and ψ_i^s, for $i=1,2,3$, may have negative, zero or positive real parts.

The coefficients for characterizing the local dynamics are:

$$\begin{aligned}\kappa_2 &= \delta^*\frac{(\rho_1+\delta)(\delta^*+\rho_1+\gamma)}{u_{cc}}\left(V_{cc}+\frac{\rho_1}{(\rho_1+\gamma)(\delta^*+\rho_1+\gamma)}\xi_{cc}\right)+\\ &\quad +(E^{'}-\delta^*)(E^{'}+P^{'})-E^{'}P^{'}+\frac{u_{NN}}{A^2u_{cc}}+\frac{\xi_cE^{''}}{Au_{cc}}-\frac{2(\rho_2\xi_{Nc}-\rho_1\xi_{cN})}{Au_{cc}}\end{aligned}$$

where $P^{'} = -\frac{u_{cN}}{Au_{cc}} < 0$ is the optimal change in pollution for increases in the natural capital stock. If $E^{'} + P^{'} < 0$ and $V_{cc} < 0$ then $\kappa_2 > 0$.

$$\begin{aligned}
\kappa_1 \quad = \quad & \rho_2(\delta^* + \rho_2)\kappa_2 + \frac{\rho_2(\delta^* + \rho_2)\xi_{NN}}{A^2 u_{cc}} + \\
& + (\rho_2(\delta^* + \rho_2) + (\rho_1 + \gamma)(\delta^* + \rho_1 + \gamma)) \Big((E^{'} - \delta^*)(E^{'} + P^{'}) - E^{'} P^{'} + \\
& + \frac{u_{NN}}{A^2 u_{cc}} + \frac{\xi_c E^{''}}{Au_{cc}} - \frac{2(\rho_2 \xi_{Nc} - \rho_1 \xi_{cN})}{Au_{cc}} \Bigg) + \\
& + \frac{\rho_2 \xi_{Nc}}{Au_{cc}} \Big((E^{'} - \delta^*)(\delta^* + \rho_2) + E^{'} \rho_2 + 2(\rho_1 + \gamma)(\delta^* + \rho_1 + \gamma) \Big) + \\
& + \frac{\rho_1 \xi_{cN}}{Au_{cc}} \Big((E^{'} - \delta^*)(\rho_1 + \gamma) + E^{'}(\delta^* + \rho_1 + \gamma) - 2\rho_2(\delta^* + \rho_2) \Big) .
\end{aligned}$$

$$\begin{aligned}
\det(J) \quad = \quad & - \frac{\delta^*(\rho_1 + \delta)(\delta^* + \rho_1 + \gamma)(\delta^* + \rho_2)}{Au_{cc}} \Bigg\{ \frac{\xi_c E^{''}}{\delta^*} + \\
& + E^{'} \Bigg[A(E^{'} - \delta^*) \left(V_{cc} + \frac{\rho_1}{(\rho_1 + \gamma)(\delta^* + \rho_1 + \gamma)} \xi_{cc} \right) + \\
& \left(V_{cN} + \frac{\rho_1}{(\rho_1 + \gamma)(\delta^* + \rho_1 + \gamma)} \xi_{cN} \right) \Bigg] + \\
& (E^{'} - \delta^*) \left(V_{Nc} + \frac{1}{(\delta^* + \rho_2)} \xi_{Nc} \right) + \left(V_{NN} + \frac{1}{(\delta^* + \rho_2)} \xi_{NN} \right) \Bigg\}
\end{aligned}$$

[`incomplete`]

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