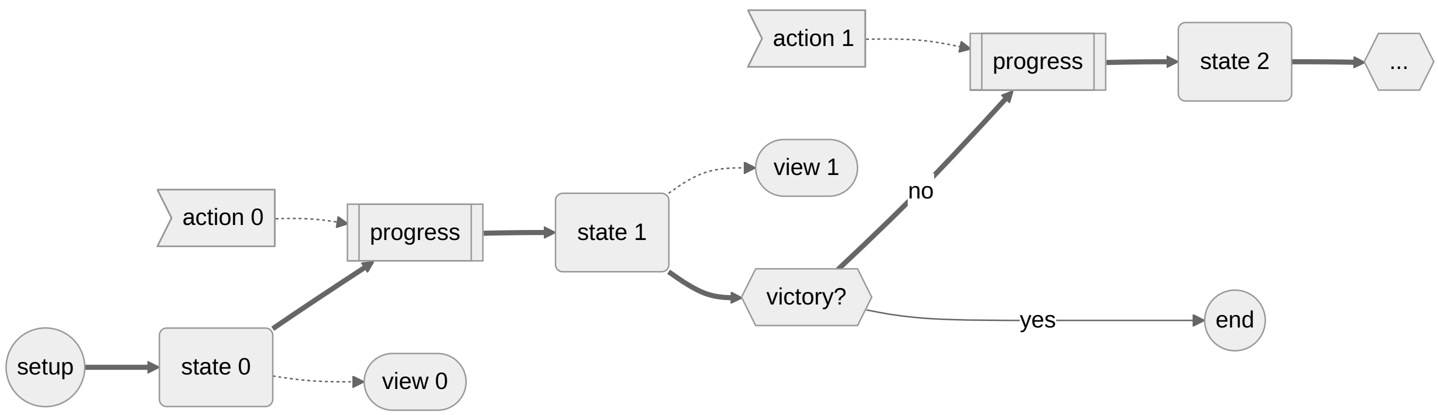
Formal Game Description

> We want to provide a **formal**, **mathematical** description of the key elements in a game:
>
> - Game State
> - Game View
> - Game Setup
> - Player Actions
> - Victory Conditions
> - Progression of Play



Game State

> The essential information about the game.

- The **set of players** is $P = \{-1, 1\}$.
- The **set of boards** is $B = \{-1, 0, 1\}^{3\times 3}$ the set of 3×3 matrices with entries from $\{-1, 0, 1\}$.
- The **set of states** is

$$S = P \times B = \{(p, b) : p \in P, b \in B\}$$

- If $p \in P, b \in B$ we also define the following functions:

$$
\begin{aligned}
s = (p, b) &= state(p, b) \in S && \text{create state} \\
p &= player(s) && \text{get player} \\
b &= board(s) && \text{get board} \\
place(b, x, i, j) &= b' \in B \text{ such that } \begin{cases} b'_{ij} &= x \\ b'_{mn} &= b_{mn} \text{ for } (m, n) \neq (i, j) \end{cases} && \text{action outcome} \\
next(p) &= -p && \text{next player} \\
col(s, j) &= \{board(s)_{1j}, board(s)_{2j}, board(s)_{3j}\} && \text{column } j \\
row(s, i) &= \{board(s)_{i1}, board(s)_{i2}, board(s)_{i3}\} && \text{row } i \\
dd(s) &= \{board(s)_{11}, board(s)_{22}, board(s)_{33}\} && \text{desc. diagonal} \\
ad(s) &= \{board(s)_{13}, board(s)_{22}, board(s)_{31}\} && \text{asc. diagonal}
\end{aligned}
$$

Game View

> What part of a state s is shown to player q?

If $s \in S, q \in P$:

$$view(s, q) = s$$

Other games might show only some part of the *full* state. For example, imagine a card game; the state describes each player's hands plus the deck and the stack; but, **for each player**, only her's own hand and possibly the cards on the desk are *visible*.

Game Setup

> How does the game starts, *i.e.*, what is the initial state?

$$setup() = s_0 = \left(1, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

Player Actions

> What are the players actions and how do they define the next state.

The effect of action i, j by player q in state s is defined by:

$$action(s,q,i,j) = \begin{cases} s & \text{if} \quad q \neq player(s) \vee \\ & \quad i < 0 \vee i > 3 \vee j < 0 \vee j > 3 \vee \\ & \quad board(s)_{ij} \neq 0 \\ (next(q), place(board(s), q, i, j)) & \text{otherwise} \end{cases}$$

> *Legal* actions define a new state while *illegal* ones keep the state unchanged.

Victory Conditions

> When the player q wins the game.

$$victory(s,q) \Leftrightarrow \begin{cases} q \neq 0 \quad \wedge \\ \Big(\quad ad(s) = \{q\} \vee \\ \quad dd(s) = \{q\} \vee \\ \quad col(s,1) = \{q\} \vee \quad col(s,2) = \{q\} \vee \quad col(s,3) = \{q\} \vee \\ \quad row(s,1) = \{q\} \vee \quad row(s,2) = \{q\} \vee \quad row(s,3) = \{q\} \quad \Big) \end{cases}$$

Here we also define a **draw state** as:

$$draw(s) \Leftrightarrow \neg victory(s,1) \wedge \neg victory(s,-1) \wedge \forall i,j \in \{1,2,3\}, board(s)_{ij} \neq 0.$$

> **Note** that *e.g.* $col(s,3) = \{q\}$ states that, in state s, *all the elements in the third column are equal to* q.

Progression of Play

Given a state $s \in S$, the progress (next state) that results from action i, j by player q is:

$$progress(s,q,i,j) = \begin{cases} action(s,q,i,j) & \text{if} \quad \neg victory(s, next(q)) \wedge \neg draw(s) \\ s & \text{otherwise} \end{cases}$$

> The state only progress if it is not a *ending state* and the action is *legal*.