SDE harvesting models in random environments: The effect of Allee effects, model robust properties and profit optimization

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Abstract

We seek model robust properties, so we work with a general function $f \in C^1 :]0, +\infty[\mapsto \mathbb{R}$, which, besides mild technical assumptions (like having left limit $f(0^+)$ and $\lim_{X \to 0^+} X f(X) = 0$), satisfies only qualitative conditions dictated by biological considerations. Since one main purpose is to study the effect of Allee effects, we compare general models without Allee effects (f strictly decreasing with $f(0^+) > 0$ and $f(+\infty) < 0$, which implies the existence of a carrying capacity $K > 0$ such that $f(K) = 0$) with general models with Allee effects (there is an $L \in]0, K[$ such that $f(L) > 0$, $f(K) = 0$, $f(X)$ decreases for $X > L$, but, due to the Allee effects, $f(X)$ increases for $0 < X < L$).

Considering autonomous efforts $E(t) = E(X(t))$ and defining the **geometric average net growth rate** $n(X) = g(X) - qE(X)$ (difference between the geometric average natural growth rate $g(X) = f(X) - \sigma^2/2$ and the harvesting mortality rate $qE(X)$), the deciding factor is the sign of its limit $n(0^+)$ at low population sizes. If the sign is negative (overfishing), we have population extinction. If the sign is positive, we have sustainability, with $X(t)$ ergodic converging as $t \to +\infty$ to a stochastic equilibrium with a stationary density function proportional to the speed density. That was shown in [1, 2, 3] for general models without Allee effects and in [13] for general Allee effects models without harvesting (i.e. with $E(x) \equiv 0$). We have now proved it for general Allee effects models with autonomous harvesting.

However, realistic extinction (in the sense of population droping below some small positive extinction threshold $a < X(0)$) occurs in both cases. Expressions for the mean and standard deviation of extinction times can be seen in [12, 4] and [11] uses them to determine the influence of Allee effects on extinction.

When applying to real data to numerically compare harvesting policies and their associated profits, we do obviously work with specific models, namely the logistic model $f(X) = r(1 - \frac{X}{K})$ ($r > 0$), which has no Allee effects, and the logistic-like model with Allee effects $f(X) = r(1 - \frac{X}{K})(\frac{X-A}{K-A})$ (with $A \in] - K, 0[$, i.e. with weak Allee effects since strong Allee effects lead to extinction, even without harvesting).

In previous work [5], for the logistic model and using data from [14] on the Pacific halibut (*Hippoglossus hippoglossus*), we have shown that the harvesting policy with variable effort is inapplicable, whereas the optimal harvesting policy with constant effort $E(t) \equiv E$ is easily applicable and leads to population sustainability, with only a slightly lower profit. So, we have also considered [6, 7, 9] stepwise policies, which are applicable but share some of the problems of the optimal variable effort policy, penalized profit optimal policies (with an artificial running energy cost on the effort), which eliminate some of the disadvantages but are still inapplicable, and combinations of these policies.

We also applied these policies to the logistic-like model with Allee effects [8, 10] to study the influence of Allee effects and check whether they should be taken into account when designing harvesting policies.

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