# Riemannian geometry of the twistor space of a symplectic manifold

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## 0.1 The metric

In this short communication we show some computations about the curvature of a metric defined on the twistor space of a symplectic manifold.

Let $(M,\omega,\nabla)$ be a symplectic manifold endowed with a symplectic connection (that is $\nabla\omega=0,\ T^\nabla=0$). Recall that the twistor space

$$\mathcal{Z}=\big\{j\in\mathrm{End}\,T_xM:\ x\in M,\ j^2=-1,\ \omega\text{ type (1,1) for } j \text{ and } \omega(\ ,j\ )>0\big\}$$

is a bundle $\pi:\mathcal{Z}\rightarrow M$, with obvious projection, together with an almost complex structure $\mathcal{J}^\nabla$ defined as follows. First, notice the connection induces a splitting

$$0\longrightarrow\mathcal{V}\longrightarrow T\mathcal{Z}=\mathcal{H}^\nabla\oplus\mathcal{V}\xrightarrow{\mathrm{d}\pi}\pi^*TM\longrightarrow 0$$

into horizontal and vertical vectors, which is to be preserved by $\mathcal{J}^\nabla$. Since the fibres of $\mathcal{Z}$ are hermitian symmetric spaces $Sp(2n,\mathbb{R})/U(n)$ — the Siegel domain —, we may identify

$$\mathcal{V}_j=\{A\in\mathfrak{sp}(\pi^*TM,\pi^*\omega):\ Aj=-jA\}$$

and hence $\mathcal{J}^\nabla_j$ acts like left multiplication by $j$: $\mathcal{J}^\nabla_j(A)=jA$.

On the horizontal part, the twistor almost complex structure is defined in a tautological fashion as $j$ itself, up to the bundle isomorphism $\mathrm{d}\pi_| :\mathcal{H}^\nabla\rightarrow\pi^*TM$ which occurs pointwise: thus $\mathcal{J}^\nabla_j(X)=(\mathrm{d}\pi)^{-1}j\mathrm{d}\pi(X),\ \ \forall X$ horizontal.

Notice that we understand that $j\in\mathcal{Z}$ also belongs to $\mathrm{End}\,(\pi^*TM)_j$, so there exists a canonical section $\Phi$ of the endomorphisms bundle defined by $\Phi_j=j$.

In [1,2] a few properties and examples of this twistor theory are explored. Between them, the integrability equation is recalled (cf. [3]), dependent on the curvature of $\nabla$ only. A natural hermitian metric on $\mathcal{Z}$ was also considered in [1,2] and our aim now is to find the sectional curvature in a special case. First we define its associated non-degenerate 2-form $\Omega^\nabla$. By analogy with the Killing form in Lie algebra theory and a Cartan's decomposition of $\mathfrak{sp}(2n,\mathbb{R})=\mathfrak{u}(n)\oplus\mathfrak{m}$, the subspace $\mathfrak{m}$ playing the role of $\mathcal{V}_j$, one defines a symplectic form on $\mathcal{Z}$ by $\Omega^\nabla = t\,\pi^*\omega-\tau$, where $t\in]0,+\infty[$ is a fixed parameter and

$$\tau(X,Y) \;=\; \frac{1}{2}\mathrm{Tr}\,(PX)\Phi(PY).$$

$P$ is the projection $T\mathcal{Z}$ onto $\mathcal{V}$ with kernel $\mathcal{H}^\nabla$, thus a $\mathcal{V}$-valued 1-form on $\mathcal{Z}$. It is easy to see that $\mathcal{J}^\nabla$ is compatible with $\Omega^\nabla$ and that the induced metric is positive definite. The following results are proved in the cited thesis.

**Theorem 0.1.** *$\Omega^\nabla$ is closed iff $\nabla$ is flat. In such case, $\mathcal{Z}$ is a Kähler manifold.*

Let $\langle\ ,\ \rangle$ be the induced metric, so that

$$\langle X,Y\rangle \;=\; t\,\pi^*\omega(X,\mathcal{J}^\nabla Y)+\frac{1}{2}\mathrm{Tr}\,(PXPY)$$

and thus $\mathcal{H}^\nabla\perp\mathcal{V}$.

**Lemma 0.1.** *$P$ is a $\mathcal{V}\subset\mathrm{End}\,(\pi^*TM)$-valued 1-form on $\mathcal{Z}$. The connection $D=\pi^*\nabla-P$ on $\pi^*TM$ preserves $\mathcal{V}$ and hence induces a new linear connection $D$ over the twistor space such that $D\mathcal{J}^\nabla=0$ and $D$ preserves the splitting of $T\mathcal{Z}$. Moreover, the torsion $T^D=P(\pi^*R^\nabla)-P\wedge\mathrm{d}\pi$.*

Let $\cdot^h$ denote the horizontal part of any tangent-valued tensor.

**Theorem 0.2.** *(i) The Levi-Civita connection of $\langle\ ,\ \rangle$ is given by*

$$\mathfrak{D}_XY \;=\; D_XY-PY(\pi_*X)-\frac{1}{2}P(\pi^*R^\nabla_{X,Y})+S(X,Y)$$

*where $S$ is symmetric and defined both by*

$$\langle P(S(X,Y)),A\rangle \;=\; \langle A\pi_*X,\pi_*Y\rangle,\qquad \forall A\in\mathcal{V},$$

*and*

$$\langle S^h(X,B),Y\rangle \;=\; \frac{1}{2}\langle P(\pi^*R^\nabla_{X,Y}),B\rangle,\qquad \forall Y\in\mathcal{H}^\nabla.$$

*Hence for $X,Y\in\mathcal{H}^\nabla$ and $A,B\in\mathcal{V}$ we have*

$$P(S(X,A))=P(S(A,B))=0,$$
$$S^h(X,Y)=S^h(A,B)=0.$$

*(ii) The fibres $\pi^{-1}(x)$, $x\in M$, are totally geodesic in $\mathcal{Z}_M$.*
*(iii) If $\nabla$ is flat, then $\mathfrak{D}\mathcal{J}^\nabla=0$.*

One may write $P(S(X,Y))$ explicitly and construct a symplectic-orthonormal basis of $\mathcal{V}$ induced by a given such basis on $\mathcal{H}^\nabla$. We show the first of these assertions.

**Proposition 0.1.** *For $X,Y$ horizontal*

$$S_j(X,Y) \;=\; -\frac{t}{2}\Big\{\omega(X,\ \ )jY+\omega(jY,\ \ )X+\omega(jX,\ \ )Y+\omega(Y,\ \ )jX\Big\}.$$

*In particular,* $\langle S_j(X,Y)X,Y\rangle=\frac{1}{2}\big(\langle X,Y\rangle^2+\|X\|^2|Y\|^2+t^2\omega(X,Y)^2\big)$ *and* $\langle S_j(X,X)Y,Y\rangle=\langle X,Y\rangle^2-t^2\omega(X,Y)^2$.

The proof of the last result is accomplished by simple verifications. The following is the relevant linear algebra used in its discovery, explained to us by J. Rawnsley. Since $\mathfrak{sp}(2n,\mathbb{R})\simeq S^2\mathbb{R}^{2n}$, the symmetric representation space, which is irreducible under $Sp(2n,\mathbb{R})$, and since

$$\omega^2(XY,ZT)=\omega(X,Z)\omega(Y,T)+\omega(X,T)\omega(Y,Z)$$

is a non-degenerate symmetric bilinear form, it follows that $\omega^2$ must be a multiple of the Killing form of $\mathfrak{sp}$, ie. the trace form!

The twistor space is not compact, nor does the metric extend to any compact space that we know. Indeed, we have not yet found a proof for the following **conjecture**: if $\nabla$ is complete, the same is true for $D$ and $\mathfrak{D}$.

## 0.2 Kählerian twistor spaces

The next result appeared in [1] without a proof. Until the end of the subsection assume $R^\nabla=0$, ie. that the metric $\langle\ ,\ \rangle$ is Kählerian.

**Theorem 0.3.** *Let $\Pi$ be a 2-plane in $T_j\mathcal{Z}$ spanned by the orthonormal basis $\{X+A,Y+B\}$, $X,Y\in\mathcal{H}^\nabla$, $A,B\in\mathcal{V}$. Then the sectional curvature of $\Pi$ is*

$$\begin{aligned} k_j(\Pi) &= -\langle R^{\mathfrak{D}}(X+A,Y+B)(X+A),Y+B\rangle\\ &= \tfrac{1}{2}\Big(\|X\|^2\|Y\|^2+3t^2\omega(X,Y)^2-\langle X,Y\rangle^2\Big)+\\ &\qquad +\|BX-AY\|^2-2\langle[B,A]X,Y\rangle-\|[B,A]\|^2\end{aligned}$$

*where $[\,,\,]$ is the commutator bracket. Thus*

$$k_j(\Pi)\begin{cases} >0 & \text{for } \Pi\subset\mathcal{H}^\nabla\\ <0 & \text{for } \Pi\subset\mathcal{V}\ .\end{cases}$$

*Proof.* Following the previous theorem, notice that $S$ is vertical only. Let $U,V$ be any two tangent vector fields over $\mathcal{Z}$. Then

$$\begin{aligned} \mathrm{d}^{\pi^*\nabla}P(U,V) &= \pi^*\nabla_U(PV)-\pi^*\nabla_V(PU)-P[U,V]\\ &= D_UPV+[PU,PV]-D_VPU-[PV,PU]-P[U,V]\\ &= PT^D(U,V)+2[PU,PV]\;=\;2[PU,PV].\end{aligned}$$

Hence, from well known connection theory,

$$R^D = R^{\pi^*\nabla} - \mathrm{d}^{\pi^*\nabla} P + P\wedge P \;=\; -P\wedge P.$$

Now let us use the notation $\mathcal{R}_{uvwz} = \langle R^{\mathfrak{D}}(U,V)W,Z\rangle$. Recall the symmetries $\mathcal{R}_{uvwz} = \mathcal{R}_{wzuv} = -\mathcal{R}_{uvzw}$ and Bianchi identity $\mathcal{R}_{uvwz}+\mathcal{R}_{vwuz}+\mathcal{R}_{wuvz}=0$. Now we want to find

$$\begin{aligned}
-k_j(\Pi) &= \langle R^{\mathfrak{D}}(X+A,Y+B)(X+A),Y+B\rangle\\
&= \mathcal{R}_{xyxy}+\mathcal{R}_{xyxb}+\mathcal{R}_{xyay}+\mathcal{R}_{xyab}\\
&\quad +\mathcal{R}_{xbxy}+\mathcal{R}_{xbxb}+\mathcal{R}_{xbay}+\mathcal{R}_{xbab}\\
&\quad +\mathcal{R}_{ayxy}+\mathcal{R}_{ayxb}+\mathcal{R}_{ayay}+\mathcal{R}_{ayab}\\
&\quad +\mathcal{R}_{abxy}+\mathcal{R}_{abxb}+\mathcal{R}_{abay}+\mathcal{R}_{abab}
\end{aligned}$$

and, if we see this sum as a matrix, then we deduce that it is symmetric.

Notice that $R^{\mathfrak{D}}(X,Y)Z$, with $X,Y,Z$ horizontal, and $R^{\mathfrak{D}}(A,B)C$, with $A,B,C$ vertical, can be obtained immediately from Gauss-Codazzi equations. First, notice that the horizontal distribution is integrable when $\nabla$ is flat. Then the horizontal leaves are immediately seen to have $D$, or simply $\pi^*\nabla$, for Levi-Civita connection with the induced metric; hence they are flat. Finally, $S$ is the 2nd fundamental form, so a formula of Gauss says $R^{\mathfrak{D}}_{X,Y}Z = R^{\pi^*\nabla}_{X,Y}Z + S(X,Z)Y - S(Y,Z)X$. Therefore

$$\begin{aligned}
-\mathcal{R}_{xyxy} &= \langle S(X,Y)X,Y\rangle - \langle S(X,X)Y,Y\rangle\\
&= \tfrac12\big(\langle X,Y\rangle^2 + \|X\|^2|Y\|^2 + t^2\omega(X,Y)^2\big) - 2\langle X,Y\rangle^2 + 2t^2\omega(X,Y)^2\big)\\
&= \tfrac12\big(\|X\|^2|Y\|^2 + 3t^2\omega(X,Y)^2 - \langle X,Y\rangle^2\big).
\end{aligned}$$

which is positive, as we have deduced following proposition 0.1.

By the same principles, $R^{\mathfrak{D}}_{A,B}C = R^D_{A,B}C = [-P\wedge P(A,B),C] = -[[A,B],C]$. For the (totally geodesic) vertical fibres of $\mathcal{Z}$, we recall that $\mathcal{R}_{abab} = -\langle[[A,B],A],B\rangle = \|[B,A]\|^2$ is minus the sectional curvature of the hyperbolic space $Sp(2n,\mathbb{R})/U(n)$. We also note that the previous curvatures return, respectively, to the horizontal and vertical subspaces. Hence we get

$$\mathcal{R}_{xyxb} = \mathcal{R}_{xyay} = \mathcal{R}_{xbab} = \mathcal{R}_{ayab} = 0.$$

Now we want to find $\mathcal{R}_{xbay}$. First we deduce via theorem 0.2 the formulae $\mathfrak{D}_A X = D_A X$, $\mathfrak{D}_X A = D_X A - AX$, $\mathfrak{D}_A B = D_A B$. Also, the Lie bracket $[X,B] = D_X B - D_B X - T^D(X,B) = D_X B - D_B X - BX$ by lemma 0.1. Thus

$$\begin{aligned}
R^{\mathfrak{D}}_{X,B}A &= \mathfrak{D}_X\mathfrak{D}_B A - \mathfrak{D}_B\mathfrak{D}_X A - \mathfrak{D}_{[X,B]}A\\
&= \mathfrak{D}_X D_B A - \mathfrak{D}_B D_X A + \mathfrak{D}_B(AX) - \mathfrak{D}_{D_XB-D_BX-BX}A\\
&= D_X D_B A - (D_B A)X - D_B D_X A + D_B(AX) - D_{[X,B]}A - A(D_B X) - ABX\\
&= R^D_{X,B}A - ABX \;=\; -ABX.
\end{aligned}$$

Hence $\mathcal{R}_{xbay} = -\langle ABX,Y\rangle$, $\mathcal{R}_{xbxb} = \langle B^2X,X\rangle = -\|BX\|^2$ and

$$\mathcal{R}_{xyab} = \mathcal{R}_{abxy} = -\mathcal{R}_{xaby} - \mathcal{R}_{bxay} = \langle BAX,Y\rangle - \langle ABX,Y\rangle = \langle [B,A]X,Y\rangle.$$

Finally

$$\begin{aligned}
k_j(\Pi) &= -\mathcal{R}_{xyxy} - 2\mathcal{R}_{xyab} - \mathcal{R}_{xbxb} - 2\mathcal{R}_{xbay} - \mathcal{R}_{ayay} - \mathcal{R}_{abab} \\
&= -\mathcal{R}_{xyxy} + 2\langle [A,B]X, Y\rangle + \|BX\|^2 + 2\langle ABX, Y\rangle + \|AY\|^2 - \mathcal{R}_{abab} \\
&= -\mathcal{R}_{xyxy} + 2\langle [A,B]X, Y\rangle + \|BX - AY\|^2 - \mathcal{R}_{abab}
\end{aligned}$$

as we wished. The second part of the result follows by Cauchy inequality. ■

It is possible to prove that the sectional curvature attains the value $-4$ in vertical planes and a the maximum value 2 in horizontal planes. The following problem is closely related to this.

## 0.3 A problem in variational calculus

Let $T$ be a real vector space. Let $R$ be a Riemannian curvature-type tensor, ie. an element of $\bigwedge^2 T^* \otimes \bigwedge^2 T^*$ satisfying Bianchi identity and $R(u,v,z,w) = R(z,w,u,v)$. Let

$$k : Gr(2,T) \to \mathbb{R}$$

be the induced sectional curvature function on the real Grassmannian of 2-planes of $T$. Let $H \oplus V$ be a direct sum decomposition of $T$ and suppose $k$ is positive in $H$ and negative in $V$. Then, are the maximum and minimum of $k$, respectively, in $H$ and $V$?

We do not know a reference for this result — which we believe to be true. We thank any comments or guidance to the related literature.

# References


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2. R. Albuquerque and J. Rawnsley. Twistor theory of symplectic manifolds. submitted.

3. N. O'Brian and J. Rawnsley. Twistor spaces. *Annals of Global Analysis and Geometry*, 3(1):29–58, 1985.