Invariant connections on Euclidean space

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Abstract

We recall and solve the equivalence problem for a flat C^1 connection ∇ in Euclidean space, with methods from the theory of differential equations. The problem consists in finding an affine transformation of $\mathbb{R}^n$ taking ∇ to the so called trivial connection. Generalized solutions are found in dimension 1 and some specific problems are solved in dimension 2, mainly dealing with flat connections. A description of invariant connections in the plane is attempted, in view the study of real orbifolds. Complex meromorphic connections are shown in the cone $cL(p,q)$ of a lens-space.

Key Words: linear connection, invariant connection, equivalence problem, orbifold.

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1 Introduction

We wish to study linear connections ∇ on $\mathbb{R}^n$ which are less than smooth from the point of view of the differentiable class, i.e. their Christoffel symbols are not C^∞. We are particularly interested in observing the behaviour of the associated tensors, such as the torsion and curvature, and solving the *equivalence problem* in the framework of non-smooth connections on smooth manifolds. This is generically as follows: given two manifolds M_1, M_2 endowed with linear connections ∇_1, ∇_2, when does there exist a diffeomorphism $\Phi : M_1 \to M_2$ such that $\Phi \cdot \nabla_1 = \nabla_2$. The diffeomorphism is then called an affine transformation.

The equivalence problem is solved in [11] in the category of analytic manifolds with analytic connections, so it seems that the problem should be undertaken with PDE tools. Under mild conditions, we solve it for the case of the trivial connection in $\mathbb{R}^n$, leaving aside the demand of analyticity. For one particular example in $\mathbb{R}^2$ we explicitly give the solution.

We also recall invariant connections for some groups of diffeomorphisms, i.e. groups of affine transformations for a same ∇. These are most relevant in the theory of symmetric spaces. Translations plus one isomorphism F invariant ∇ are studied in $\mathbb{R}^2$, in order to bring curvature and holonomy issues into the theory of orbifolds.

The equivalence problem is an old theme, as we may see e.g. in [9–11], yet its importance in geometry remains.

Orbifolds are a generalization of manifolds to include the notion of singularities of the kind of $\mathbb{R}^n/G$ in 0, where G is a finite subgroup of GL_n. Certainly any definition of connection in this new category will agree locally with a G-invariant connection in Euclidean space (cf. [7, 8]).

We have shown that $\mathbb{R}^2/\langle F\rangle$, where F is the conjugation map, admits a symplectic connection, torsion free, with non-vanishing curvature. Also we prove all foldings by conjugate-rotations of the plane $F(z) = e^{i\theta}\overline{z}$ admit some specific flat non-trivial connections. We have looked for translation invariant connections since they are easier to find. Though we should leave the translations invariance dependence, to have freedom in coordinates so that the question of which connection-curvature really interprets the orbifold singularity may be well posed.

To finish this article on the quest towards local invariant connections, we treat the case of lens spaces $L(p, q)$ and their cone singularity. Here the case is of meromorphic objects and, indeed, we find a family of such conections, non-flat. We remark this new ∇ is just an unnoticed particular case within the whole subject of [12].

1.1 Linear connections

Let M be any paracompact smooth manifold and let $\mathfrak{X}_U$ denote the Lie algebra of smooth vector fields on an open subset U of M.

We recall the notion of a *linear connection* on a manifold M. It is given by a *covariant derivative*, i.e. an operator ∇ on the space of pairs of smooth

vector fields X, Y defined on M, sending another smooth vector field $\nabla_X Y$ on M, and satisfying the following relations:
(i) $\nabla_X(fY) = \mathrm{d}f(X)Y + f\nabla_X Y$ (called the *Leibniz identity*),
(ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$,
(iii) $\nabla_{fX_1+X_2}Y = f\nabla_{X_1}Y + \nabla_{X_2}Y$, $\forall f \in C^\infty_M,\ \forall X, X_1, X_2, Y, Z \in \mathfrak{X}_M$.

From the first two conditions it follows that ∇ is a *local* operator (cf. [6]): if two vector fields Y_1, Y_2 agree on some open subset U, then so do their covariant derivatives. To see this suppose $Y = 0$ on U, then for each point $x \in U$ take a function $f \in C^\infty_M$ with $\operatorname{supp} f \subset U$ and $f = 1$ on a neighborhood of x (these functions exist always). Then $fY = 0$ on M, so

$$0 = \nabla 0_{\,|x} = f(x)\nabla Y_{\,|x} + \mathrm{d}f(\cdot)Y_x = \nabla Y_{\,|x}$$

and hence $(\nabla Y)_{|U} = 0$, as wished. Of course the same proof applies on the X variable.

Now by taking extensions of vector fields we may define the covariant derivative of vector fields defined only on some open subset U. The resulting vector field on U does not depend on the chosen extension. Moreover ∇ commutes with restrictions, i.e. if $V \subset U$ is another open subset, then $\nabla_X Y_{|V} = \nabla_{X_{|V}} Y_{|V}$.

Contrary to other local operators, as for instance the Lie bracket of vector fields, the covariant derivative of $Y \in \mathfrak{X}_U$ induces a well-defined linear map $\nabla Y : T_m M \to T_m M$ for any $m \in U$; for each $v \in T_m M$ just take a chart $(x_1, \ldots, x_n)$ around m and any smooth functions f_i such that $v = X_m$, where $X = \sum f_i \frac{\partial}{\partial x_i}$. Then the previous facts and condition (iii) imply that we can define $\nabla_v Y := \nabla_X Y_{\,|m} = \sum f_i(m)\nabla_{\frac{\partial}{\partial x_i}} Y_{\,|m}$ — which therefore does not depend nor on the chart, nor on the extension X of v.

The following two tensors are used in the study of linear connections. The *torsion*

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

and the *curvature*

$$R^\nabla(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

These are tensors indeed, linear over the C^∞_U ring, as it is easy to prove. One can see the curvature as a measure of how covariant derivatives of Z commute, along the directions X, Y, up to the one along the Lie commutator of X and Y. The connection is called *flat* if $R^\nabla = 0$. Obviously, $T^\nabla \in \Omega^2(TM)$ and $R^\nabla \in \Omega^2(\operatorname{End} TM)$.

Connections determine geometry of manifolds by their ability to induce *parallel displacement*. In the tangent bundle they also give the notion of *geodesics*, i.e. curves γ which satisfy $\nabla_{\gamma'}\gamma' = 0$ (we may deduce as above that the operator $\nabla_{\gamma'}$ is well defined over a curve γ, i.e., if $Y, \tilde{Y}$ are vector fields on a neighborhood of γ such that $Y_\gamma = \tilde{Y}_\gamma$, then $\nabla_{\gamma'}Y = \nabla_{\gamma'}\tilde{Y}$).

To finish, suppose we have two connections ∇^1, ∇^2. Then it is trivial to check that their difference is a tensor: $\nabla^1 = \nabla^2 + \Gamma$ with $\Gamma_X \in \operatorname{End} TM$, $\forall X \in TM$, or simply $\Gamma \in \Omega^1(\operatorname{End} TM)$.

1.2 Diffeomorphisms action on connections

We recall here other well known facts about connections.

Let M, N be two manifolds and suppose $\Phi : M \to N$ is a smooth diffeomorphism. Then Φ induces a linear map $X \mapsto \Phi \cdot X$ defined by:

$$\Phi \cdot X_{\ y} = \mathrm{d}\Phi(X_{\Phi^{-1}(y)}), \qquad \forall y \in N.$$

Proposition 1.1. $\Phi : \mathfrak{X}_M \to \mathfrak{X}_N$ *is a Lie algebra homomorphism.*

Proof. We just have to evaluate the action of the Lie bracket on smooth functions. Let $f \in C^\infty_N$. Then $(\Phi\cdot X)(f)_y = \mathrm{d}f(\Phi\cdot X_{\ y}) = \mathrm{d}(f\circ\Phi)(X_{\Phi^{-1}(y)}) = X(f \circ \Phi)_{\Phi^{-1}(y)}$. So

$$(\Phi \cdot X)(f) = X(f \circ \Phi) \circ \Phi^{-1}. \tag{1.1}$$

Hence for two vector fields on M

$$(\Phi \cdot X)((\Phi \cdot Y)(f)) = X((\Phi \cdot Y)(f) \circ \Phi) \circ \Phi^{-1} = X(Y(f \circ \Phi)) \circ \Phi^{-1}.$$

Now it is easy to see that $(\Phi \cdot [X, Y])(f) = [\Phi \cdot X, \Phi \cdot Y](f)$. ∎

Notice that, for any $h \in C^\infty_M$, we have $\Phi \cdot (hX) = (h \circ \Phi^{-1})\Phi \cdot X = (\Phi \cdot h)\Phi \cdot X$, extending notation to functions. Also notice that formula (1.1) can be written as $(\Phi \cdot X)(\Phi \cdot h) = \Phi \cdot (X(h))$.

Given a diffeomorphism $\Psi : N \to O$ to any other manifold O, we have $\Psi \cdot (\Phi \cdot X) = (\Psi \circ \Phi) \cdot X$.

An even more surprising property of the 'push-forward' map is that it acts on the space of connections. Given a connection ∇ on M we may define a new connection $\Phi \cdot \nabla$ on N by

$$(\Phi \cdot \nabla)_Z W = \Phi \cdot \left(\nabla_{\Phi^{-1}\cdot Z}\Phi^{-1} \cdot W\right)$$

for any $Z, W \in \mathfrak{X}_N$. The only non trivial identity to check is the Leibniz identity:

$$\begin{aligned}(\Phi\cdot\nabla)_Z fW \;&=\; \Phi\cdot\big(\nabla_{\Phi^{-1}\cdot Z}\Phi^{-1}\cdot(fW)\big)\\ &=\; \Phi\cdot\big(\nabla_{\Phi^{-1}\cdot Z}(\Phi^{-1}\cdot f)(\Phi^{-1}\cdot W)\big)\\ &=\; \Phi\cdot\big((\Phi^{-1}\cdot f)\nabla_{\Phi^{-1}\cdot Z}\Phi^{-1}\cdot W+\mathrm{d}(\Phi^{-1}\cdot f)(\Phi^{-1}\cdot Z)\,(\Phi^{-1}\cdot W)\big)\\ &=\; f\Phi\cdot\big(\nabla_{\Phi^{-1}\cdot Z}\Phi^{-1}\cdot W\big)+\Phi\cdot\big((\Phi^{-1}\cdot Z)(\Phi^{-1}\cdot f)\big)W\\ &=\; f(\Phi\cdot\nabla)_Z W+Z(f)W.\end{aligned}$$

The action on connections under composition of two diffeomorphisms carries canonically, as it should: $\Psi\cdot(\Phi\cdot\nabla)=(\Psi\circ\Phi)\cdot\nabla$.

Let $\tilde{\nabla}$ be another connection on N. We recall that a map Φ which satisfies $\tilde{\nabla}=\Phi\cdot\nabla$ is called an *affine transformation*. If it is an affine transformation of M onto itself, with $\tilde{\nabla}=\nabla$, then the connection is said to be Φ *invariant*. All these definitions are in [9] or [11].

As we have been showing, any given tensors transform under the push-forward map in an obvious way. For instance $\Phi\cdot T^{\nabla}(Z,W)=\Phi\cdot(T^{\nabla}(\Phi^{-1}\cdot Z,\Phi^{-1}\cdot W))$. The following identities are easy to check:

$$T^{\Phi\cdot\nabla}=\Phi\cdot T^{\nabla},\qquad\qquad R^{\Phi\cdot\nabla}=\Phi\cdot R^{\nabla}.$$

Under affine transformations, clearly unparametrized geodesics are taken to geodesics. A map which has such a property is called a *projective transformation*. This notion has been thoroughly studied in Riemannian geometry. Recently, V. S. Matveev proved the Lichnerowicz-Obata conjecture, stating that a connected group which acts projectively on a closed Riemannian manifold, then acts affinely (cf. the proof and the history of this conjecture in [13,14]). A close question dealing with projective metric structures in real dimension 2, is found in recent [3]. Our last section studies $\mathbb{R}^2$ too.

EXAMPLE 1. The *trivial* connection d in $M=\mathbb{R}^n$ is defined as

$$\mathrm{d}_X Y=(\mathrm{d}Y_1(X),\ldots,\mathrm{d}Y_n(X))=X_1\frac{\partial Y}{\partial x_1}+\cdots+X_n\frac{\partial Y}{\partial x_n}$$

where X,Y are seen as vector-valued functions $M\to TM=M\times\mathbb{R}^n$ in the ubicuous notation $X_x=(x,X_x)$. Of course, d is torsion free and flat. Let $Diff(\mathbb{R}^n)$ denote the group of diffeomorphisms of $\mathbb{R}^n$. Then it would be interesting to know the orbit of d on the space of torsion free and flat connections, under the action $\cdot$ of any given subgoup $H\subset Diff(\mathbb{R}^n)$.

EXAMPLE 2. On homogeneous spaces $M=G/H$, one explores the use of G-invariant connections, that is, connections invariant under all left translations of M induced by elements of G. They are in 1-1 correspondence, when H is closed, with direct sum decompositions of the Lie algebra $\mathrm{Lie}(G)=\mathrm{Lie}(H)\oplus\mathfrak{m}$ such that $\mathrm{ad}\,(H)(\mathfrak{m})=\mathfrak{m}$, cf. [11].

1.3 Connections in $\mathbb{R}^n$

We shall now restrict our study to connections in Euclidean space. We change notation a bit and assume $F : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. Also we let $(x_1, \ldots, x_n)$ or $(y_1, \ldots, y_n)$ denote Euclidean coordinates and abreviate the induced vector fields $\frac{\partial}{\partial x_i}$ to ∂_i. This is just the vector e_i of the canonical basis. Writing $F(x) = y$ then

$$F \cdot \partial_{i\ y} = \mathrm{d}F(\partial_{i\ x}) = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_i}(x)\partial_{i\ y}.$$

From now on we assume Einstein's summation convention.

Let ∇ be any connection. It is determined by the *Christoffel symbols*: $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^h \partial_h$.

Proposition 1.2. *Let $\tilde{\nabla} = F \cdot \nabla$. Then the Christoffel symbols $\tilde{\Gamma}_{ij}^h$ of this new connection satisfy the equation:*

$$\frac{\partial^2 F_k}{\partial x_i \partial x_j} + \frac{\partial F_l}{\partial x_i}\frac{\partial F_m}{\partial x_j}\tilde{\Gamma}_{lm}^k \circ F = \frac{\partial F_k}{\partial x_h}\Gamma_{ij}^h. \tag{1.2}$$

Proof. We have that

$$\tilde{\nabla}_{F\cdot\partial_i} F \cdot \partial_{j\ y} = F \cdot \big(\nabla_i \partial_j\big)_{\ y} = \frac{\partial F_k}{\partial x_h}(x)\Gamma_{ij}^h(x)\partial_{k\ y}.$$

On the left hand side we have, letting $G = F^{-1}$,

$$\begin{aligned}
\tilde{\nabla}_{F\cdot\partial_i} F \cdot \partial_{j\ y} &= \tilde{\nabla}_{\frac{\partial F_l}{\partial x_i}(x)\partial_l} \frac{\partial F_m}{\partial x_j}\partial_{m\ y} = \frac{\partial F_l}{\partial x_i}(x)\tilde{\nabla}_l\Big(\frac{\partial F_m}{\partial x_j}\partial_m\Big)_{\ y} = \\
&= \frac{\partial F_l}{\partial x_i}(x)\frac{\partial^2 F_m}{\partial x_q \partial x_j}(x)\frac{\partial G_q}{\partial y_l}(y)\partial_{m\ y} + \frac{\partial F_l}{\partial x_i}(x)\frac{\partial F_m}{\partial x_j}(x)\tilde{\Gamma}_{lm}^k(y)\partial_{k\ y} = \\
&= \frac{\partial^2 F_m}{\partial x_i \partial x_j}(x)\partial_{m\ y} + \frac{\partial F_l}{\partial x_i}(x)\frac{\partial F_m}{\partial x_j}(x)\tilde{\Gamma}_{lm}^k(y)\partial_{k\ y}
\end{aligned}$$

since $G(y) = x$. Hence the formula (1.2). ■

Of course, we may write an equation analogous to (1.2) in terms of $G = F^{-1}$, since $G \cdot \tilde{\nabla} = \nabla$. Moreover, a given connection on a manifold satisfies such equation, with $\tilde{\Gamma} = \Gamma$, under any coordinate change diffeomorphism.

Given any $\tilde{\nabla}$ and ∇, when does there exist a diffeomorphism F which makes the two connections the affine transformation of one another? This is called the *equivalence problem*. In [11, chapter VI, theorem 7.4] it is proved that a solution to this problem exists locally if the connections have analytic Christoffel symbols and if higher order derivatives of the torsion and the curvature tensors satisfy

$$\phi\cdot(\nabla^k T^\nabla{}_{x_0}) = \tilde\nabla^k \tilde T^\nabla{}_{y_0}, \qquad \phi\cdot(\nabla^k R^\nabla{}_{x_0}) = \tilde\nabla^k \tilde R^\nabla{}_{y_0},$$

for all $k=0,1,2,...$ and for a linear isomorphism $\phi: T_{x_0}M\to T_{y_0}N$. Moreover, the problem is solved globally in the restricted context of analytic manifolds M,N. By a *local* solution it is meant a diffeomorphism $F:U\to V$ from a neighborhood U of x_0 onto a neighborhood V of y_0 and such that $\mathrm{d}F_{x_0}=\phi$.

Remark: An interesting consequence of this result is the following. If M is a C^∞ manifold with a C^∞ linear connection such that $\nabla T^\nabla=0,\ \nabla R^\nabla=0$, then M is an analytic manifold and the connection is analytic [11, chapter VI, theorem 7.7]. This shows that all symmetric spaces are analytic manifolds.

Remark: There is another type of transformation of linear connections we want to be aware (this applies generally to connections on vector bundles, cf. [6]). The *gauge transformations*, which we recall in the case of U open in $\mathbb{R}^n$, are defined by a map $u:U\to GL_n$ and act on connections $\nabla=\mathrm{d}+\Gamma$ almost like an "infinitesimal affine transformation" covering the identity map of the manifold. Namely, they are defined by

$$(u\nabla u^{-1})_X Y := u(\nabla_X(u^{-1}Y)) = \nabla_X Y - (\nabla_X u)u^{-1}Y$$

or, we may say, Γ transforms into $\Gamma-(\nabla u)u^{-1}$ (note that the inverse is in GL_n).

Before we proceed, we recall here in local coordinates the formula for $R^\nabla(\partial_i,\partial_j)\partial_k = R^l_{ijk}\partial_l$:

$$R^l_{ijk} = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^l_{ip}\Gamma^p_{jk} - \Gamma^l_{jp}\Gamma^p_{ik}. \tag{1.3}$$

1.4 Flat connections

It is easy to see the gauge transformation induces a conjugation by u of the curvature tensor, but the same is not true for the torsion.

Proposition 1.3. *Any flat connection ∇ is locally equal to the gauge transformation $u\mathrm{d}u^{-1}$, for some map u. Such connection is torsion free if, and only if,*

$$\sum_{l=1}^n \frac{\partial u_{ij}}{\partial x_l}u_{lk} = \sum_{l=1}^n \frac{\partial u_{ik}}{\partial x_l}u_{lj}, \qquad \forall i,j,k\in\{1,\dots,n\}. \tag{1.4}$$

Proof. Let $s_0 = (\partial_1, \ldots, \partial_n)$. We first show that there is a solution of $\nabla s = 0$, for a smooth frame $s : U \to (\mathbb{R}^n)^n$, on an open neighborhood U of each point. Writing in matrix notation $s = s_0 u$ and $\nabla s_0 = s_0 \Gamma$, we have

$$\nabla s = (\nabla s_0)u + s_0 \mathrm{d}u = s_0(\Gamma u + \mathrm{d}u).$$

Now $\Gamma u + \mathrm{d}u = 0$ is a first order linear differential equation, which has a solution if, and only if, its exterior derivative is zero. Therefore we compute $\mathrm{d}\Gamma\, u - \Gamma \wedge \mathrm{d}u + \mathrm{d}^2 u = 0$ or, equivalently, $\mathrm{d}\Gamma + \Gamma \wedge \Gamma = 0$. But, the reader may care to deduce this is the same as $R^\nabla = 0$.

The second part of the result is trivial to check. ∎

In the following, let $DF = [\frac{\partial F_i}{\partial x_j}]$. Here we state an approach to the equivalence problem for the trivial connection.

Proposition 1.4. *Let $u : \mathbb{R}^n \to GL_n$ be a C^1 map. Suppose $\nabla = u\mathrm{d}u^{-1}$ is torsion free. Then $\nabla = F \cdot \mathrm{d}$ for a diffeomorphism F if, and only if,*

$$u \circ F = DF\, k \tag{1.5}$$

with k a constant invertible matrix.

Proof. The existence of F is assumed in both cases. By straightforward computations we find $F \cdot s_0 = s_0\, DF_{|F^{-1}}$ and $(F \cdot \mathrm{d})_X(F \cdot s_0) = F \cdot (\mathrm{d}_{F^{-1} \cdot X} s_0) = 0$, with $s_0 = (\partial_1, \ldots, \partial_n)$. Hence condition (1.5), say $u = DF_{|F^{-1}}\, k$, implies that $s_0 u k^{-1}$ is parallel for $F \cdot \mathrm{d}$. As in the proof of Proposition 1.3, we know that $\nabla(s_0 u k^{-1}) = 0$ implies $F \cdot \mathrm{d} = uk^{-1}\mathrm{d}(uk^{-1})^{-1} = u\mathrm{d}u^{-1}$.

Reciprocally, if $F \cdot \mathrm{d} = u\mathrm{d}u^{-1}$ is satisfied, then $s_0 u$ is parallel for $F \cdot \mathrm{d}$. Hence $s_0 u = (F \cdot s_0)k$, for some constant k, and consequently (1.5) holds. ∎

Notice that, once we find F, we may incorporate k in u.

2 Existence results for the equivalence problem

2.1 The dimension $n = 1$ case

In $\mathbb{R}$ suppose we are given a linear connection $\nabla = \mathrm{d} + \Gamma$, with Γ a 1-form with values in $\operatorname{End}\mathbb{R} = \mathbb{R}$. Clearly, a 1-form on the real line corresponds to a function Γ^1_{11} such that $\Gamma_x(v) = \Gamma^1_{11}(x)v$, $\forall v \in \mathbb{R}$, and clearly the torsion and the curvature of ∇ both vanish. Nevertheless, we may still try to solve the equivalence problem. According to Proposition 1.2 we look for a diffeomorphism F such that (we let $\Gamma = \Gamma^1_{11}$)

$$F'' + (F')^2 \Gamma \circ F = 0. \tag{2.1}$$

Noteworthy, with the most simple non-trivial connection, that is, with Γ a non-zero constant, we obtain the transformation

$$F(x) = \frac{1}{\Gamma}\log|x+c_1| + c_2,$$

where c_1, c_2 are constants, which requires further notice on restrictions of the domain.

For the differential equation (2.1) with generic Γ, we may introduce the following weak variational problem.

Definition: We say that F, defined on an interval $]a,b[$ $(-\infty < a < b < +\infty)$, is a *generalized solution* to (2.1) if F belongs to the Sobolev space $H^1(a,b)$ and satisfies

$$F'(a)G(a) + \int_a^b F'G' = F'(b)G(b) + \int_a^b |F'|^2\Gamma(F)G, \quad \forall G \in H^1(a,b). \tag{2.2}$$

Remark: The weak formulation (2.2) is obtained by the following computation:

$$\int_a^b F''G + |F'|^2\Gamma(F)G = 0 \;\Leftrightarrow\; [F'G]_a^b - \int_a^b F'G' + \int_a^b |F'|^2\Gamma(F)G = 0.$$

When boundary conditions are taken into account, (2.1) becomes a boundary value problem.

Proposition 2.1. *Assume that Γ is a continuous real function such that*

$$\Gamma(t)t \le \alpha < 1 \quad \text{or} \quad \Gamma(t)t \ge \beta > 1, \quad \forall t \in \mathbb{R}. \tag{2.3}$$

Then the boundary value problem to (2.1) under

1. *the homogeneous Dirichlet conditions has the unique generalized solution $F \equiv 0$;*

2. *the homogeneous Neumann conditions has the generalized solutions $F \in \mathbb{R}$;*

3. *the mixed conditions, $F(a) = 0$ and $F'(b) = F_b \in \mathbb{R}\setminus\{0\}$, has a unique generalized solution F in the following sense*

$$\int_a^b F'G' = F_bG(b) + \int_a^b |F'|^2\Gamma(F)G, \quad \forall G \in V, \tag{2.4}$$

where V is the set of functions $G \in H^1(a,b)$ such that $G(a) = 0$.

Proof. Assume that $\Gamma(t)t \leq \alpha$, for all $t \in \mathbb{R}$. Otherwise the proof is analogous. Let us concentrate on the existence proof to the mixed boundary value problem (case 3) under the Galerkin method. The cases 1 and 2 are similar and simpler. Let $\mathcal{A}$ be the induced operator of the weak variational equality (2.4), i.e., $\mathcal{A} : V \to V'$ defined by

$$\langle \mathcal{A}F, G\rangle = \int_a^b F'G' - \int_a^b |F'|^2\Gamma(F)G.$$

Applying the Poincaré inequality, we recall that V is a separable Hilbert space endowed with the norm $\left(\int_a^b |G'(x)|^2 dx\right)^{1/2}$. Letting $\{H_k\}$ be a basis of V, we set the finite dimensional space as $V_N = \langle H_1, \cdots, H_N\rangle$ for $N \in \mathbb{N}$. Using (2.3) it follows that $\mathcal{A}$ is coercive:

$$\langle \mathcal{A}F, F\rangle = \int_a^b |F'|^2(1 - \Gamma(F)F) \geq (1 - \alpha)\int_a^b |F'|^2, \quad \text{with } 1 - \alpha > 0.$$

Then there exists a Galerkin solution $F_N \in V_N$ such that

$$\int_a^b F_N' G' = F_b G(b) + \int_a^b |F_N'|^2\Gamma(F_N)G, \tag{2.5}$$

for all $G \in V_N$ and, by density, for all $G \in V$. Taking $G = F_N$ in (2.5) we obtain

$$\|F_N\|_V^2 = \int_a^b |F_N'|^2 = F_b^2 + \int_a^b |F_N'|^2\Gamma(F_N)F_N \leq F_b^2 + \int_a^b |F_N'|^2\alpha.$$

Thus the Galerkin solution satisfies the estimate

$$\|F_N\|_V \leq \frac{F_b}{\sqrt{1-\alpha}}.$$

Thus we can extract a subsequence of F_N, still denoted by F_N, such that

$$\begin{array}{lll} F_N \rightharpoonup F & \text{in} & V, \\ F_N \to F & \text{in} & C([a,b]), \end{array}$$

with $F \in H^1(a,b) \hookrightarrow C([a,b])$. In order to prove that F is a solution to (2.4) we will pass to the limit in (2.5) for all $G \in V$ as N tends to infinity. To pass to the limit the term on the left hand side in (2.5), it is sufficient the weak convergence of F_N' to F' in $L^2(a,b)$. Notice that this does not allow to pass to the limit the last term on the right hand side in (2.5). So to prove the strong convergence it remains to show that

$$\|F_N'\|_{L^2} \to \|F'\|_{L^2} \quad \text{as } N \to +\infty. \tag{2.6}$$

First let us identify $|F'_N|^2$ as an element of the dual space of $C([a,b])$. Hence we can extract a subsequence of $|F'_N|^2$, still denoted by $|F'_N|^2$, weak-* convergent to χ in $L^1(a,b)$. Next passing to the limit (2.5) it results

$$\int_a^b F'G' = F_b G(b) + \int_a^b \chi\Gamma(F)G$$

for all $G \in V$. In particular taking $G = F$ we obtain

$$\int_a^b |F'|^2 = F_b^2 + \int_a^b \chi\Gamma(F)F. \tag{2.7}$$

Now passing to the limit in (2.5) when $G = F_N$ is chosen, we get

$$\lim \int_a^b |F'_N|^2 = F_b^2 + \int_a^b \chi\Gamma(F)F. \tag{2.8}$$

Finally gathering (2.7) and (2.8) we conclude (2.6). ■

2.2 The dimension $n = 2$ case

In dimension 2 we will find the integrability condition (1.4), for F belonging to C^2. As a first case to study, we present the following example.

Example 1. We consider the symmetric and flat connection $\nabla = \mathrm{d} + \Gamma$ given by $\Gamma^1_{11}(x,y) = f(x)$, $\Gamma^2_{22}(x,y) = g(y)$, where f,g are C^α, and any other $\Gamma^k_{ij} = 0$. Then ∇ is flat by trivial reasons. Solving $\nabla = F\cdot \mathrm{d}$ with $F \in C^{\alpha+2}$, implies solving for F_1 the system

$$\left\{\begin{array}{l} \partial^2_{xx}F_1 + (\partial_x F_1)^2 f(F_1) = 0 \\ \partial^2_{xy}F_1 + (\partial_x F_1)(\partial_y F_1) f(F_1) = 0 \\ \partial^2_{yy}F_1 + (\partial_y F_1)^2 f(F_1) = 0 \end{array}\right. .$$

An analogous system must be satisfied by F_2. Imposing further $\partial_y F_1 = \partial_x F_2 = 0$ we see that the problem is equivalent to solving the dimension 1 case.

Next we present a non-constant example.

Example 2. Consider the open set $\mathbb{R}^+ \times \mathbb{R}$ and a connection given by $\nabla_x\partial_x = -\frac{1}{2x}\partial_x$, $\nabla_x\partial_y = \frac{1}{2x}\partial_y = \nabla_y\partial_x$, $\nabla_y\partial_y = x\partial_x$, in real coordinate functions. An easy computation shows that ∇ is flat: $R^\nabla(\partial_x,\partial_y)\partial_x =$

$$\nabla_x\nabla_y\partial_x - \nabla_y\nabla_x\partial_x = \nabla_x\frac{1}{2x}\partial_y - \nabla_y\big(-\frac{1}{2x}\partial_x\big) = \big(-\frac{1}{2x^2} + \frac{1}{4x^2} + \frac{1}{4x^2}\big)\partial_y = 0$$

and $R^{\nabla}(\partial_x, \partial_y)\partial_y =$

$$= \nabla_x \nabla_y \partial_y - \nabla_y \nabla_x \partial_y = \nabla_x x \partial_x - \nabla_y \big(\frac{1}{2x}\partial_y\big) = \partial_x - \frac{x}{2x}\partial_x - \frac{x}{2x}\partial_x = 0.$$

Now the group-valued map u may be deduced from $\Gamma = u\mathrm{d}u^{-1} = -(\mathrm{d}u)u^{-1}$, i.e. the equations

$$\Gamma_1 u = -\frac{\partial u}{\partial x}, \qquad \Gamma_2 u = -\frac{\partial u}{\partial y}.$$

Henceforth we find that the equation in $F = (F_1, F_2)$

$$\begin{bmatrix} \frac{\sqrt{2F_1}}{2} e^{-\frac{F_2}{\sqrt{2}}} & -\sqrt{F_1} e^{\frac{F_2}{\sqrt{2}}} \\ \frac{1}{2\sqrt{F_1}} e^{-\frac{F_2}{\sqrt{2}}} & \frac{\sqrt{2}}{2\sqrt{F_1}} e^{\frac{F_2}{\sqrt{2}}} \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}$$

is the one to be solved, applying Proposition 1.4. Notice $\big\{F_1, F_2\big\} = \mathrm{Jac}\, F = 1$. This is the case where the map u takes values in $SL(2, \mathbb{R})$.

Find F_1 and F_2 in the forms $F_1(x, y) = e^{2(f(x)-g(y))}$, $F_2(x, y) = \sqrt{2}(f(x)+g(y))$; then we obtain the following equations

$$\begin{aligned} 2\sqrt{2} f'(x) e^{2f(x)-2g(y)} &= e^{-2g(y)} \\ -2g'(y) e^{2f(x)-2g(y)} &= -e^{2f(x)} \\ 2\sqrt{2} f'(x) &= e^{-2f(x)} \\ 2g'(y) &= e^{2g(y)} \end{aligned}$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{2f} = \frac{1}{\sqrt{2}}, \qquad \frac{\mathrm{d}}{\mathrm{d}y} e^{-2g} = -1.$$

Then we obtain

$$f(x) = \frac{1}{2} \log \left(\left| \frac{x}{\sqrt{2}} + c_1 \right| \right), \quad g(y) = -\frac{1}{2} \log(| - y + c_2|)$$

where c_1 and c_2 are determined according to the domain.

Considering $c_1 = c_2 = 0$, we obtain the function

$$(F_1, F_2) = \left(-\frac{xy}{\sqrt{2}}, \frac{\sqrt{2}}{2} \log(\frac{x}{\sqrt{2}(-y)}) \right), \quad x > 0, y < 0,$$

solving our particular and ilustrative problem.

Remark: In Proposition 3.3, if u is such that

$$\begin{cases} \partial_y F_1(x, y) = 0 \\ \partial_x F_2(x, y) = 0 \end{cases} \Rightarrow \begin{cases} F_1 = F_1(x) \\ F_2 = F_2(y) \end{cases}$$

and u_{11} and u_{22} are functions in F_2 and F_1, it results

$$F_1'(x) = u_{11}(x, y) \qquad F_2'(y) = u_{22}(x, y).$$

This is impossible. Then we conclude that the existence of a solution depends on u.

In conclusion, if we find (F_1, F_2) of class C^2 we have the following restrictions on u:

$$\frac{\partial}{\partial x}\left[u_{12}(F_1, F_2)\right] = \frac{\partial}{\partial y}\left[u_{11}(F_1, F_2)\right],$$
$$\frac{\partial}{\partial x}\left[u_{22}(F_1, F_2)\right] = \frac{\partial}{\partial y}\left[u_{21}(F_1, F_2)\right]$$

or, equivalently,

$$\frac{\partial u_{12}}{\partial \xi_1}u_{11} + \frac{\partial u_{12}}{\partial \xi_2}u_{21} = \frac{\partial u_{11}}{\partial \xi_1}u_{12} + \frac{\partial u_{11}}{\partial \xi_2}u_{22},$$
$$\frac{\partial u_{22}}{\partial \xi_1}u_{11} + \frac{\partial u_{22}}{\partial \xi_2}u_{21} = \frac{\partial u_{21}}{\partial \xi_1}u_{12} + \frac{\partial u_{21}}{\partial \xi_2}u_{22}.$$

2.3 In dimension n

Here we state the existence result to Proposition 1.4. In order to adapt the proof of the generalized Frobenius Theorem [15, pp. 167], we rewrite (1.5) as

$$u_{ij} = \partial_l F_i k_{lj} \quad \Leftrightarrow \quad \partial_j F_i = u_{il} w_{lj},$$

with w denoting the inverse matrix of k. Let us begin by stating the following existence result.

Proposition 2.2. *Let u be a $GL_n(\mathbb{R})$-valued function in C^1 such that*

$$\sup_{\xi\in\mathbb{R}^n} \|u_{il}(\xi) w_{lj}\| \le K_{ij} \tag{2.9}$$

$$\sup_{\xi\in\mathbb{R}^n} \left\|\frac{\partial u_{il}}{\partial \xi_p}(\xi) w_{lj}\right\| \le K_{pij} \tag{2.10}$$

and for any $\delta < 1/\max\{K_{pij}\}$ set $\bar{Q} = [-1, 1] \times \bar{B}_\delta(0) \subset \mathbb{R}^{n+1}$. Then there exists $z \in C^1(\bar{Q})$ such that

$$\partial_t z_i(t, x) = u_{il}(z(t, x)) w_{lq} x_q. \tag{2.11}$$

Moreover, the solution z verifies

$$\partial_j z_i(t, x) = \int_0^t \Big(\frac{\partial u_{il}}{\partial z_p}(z(\tau, x))\partial_j z_p(\tau, x) w_{lq} x_q + u_{il}(z(\tau, x)) w_{lj}\Big) d\tau. \tag{2.12}$$

Proof. In order to apply the Schauder fixed point theorem [15, pp. 56], let us consider the ball, with radious $R > 0$,

$$B_R := \{\xi \in C^1(\bar{Q}) : \ \|\xi\|_{C^1} \leq R\}.$$

Let us construct the mapping $\mathcal{L} : \xi \mapsto z$ as follows

$$z_i(t,x) = \int_0^t u_{il}(\xi(\tau,x)) w_{lq} x_q d\tau.$$

From (2.9), it follows

$$\max_{\bar{Q}} \|z\| \leq \max\{K_{ij}\}\delta, \quad \max_{\bar{Q}} \|\partial_t z\| \leq \max\{K_{ij}\}\delta.$$

From (2.10), the derivative of z_i with respect to x_j verifies

$$\begin{aligned} \|\partial_j z_i\| &= \left\| \int_0^t \Big(\frac{\partial u_{il}}{\partial \xi_p}(\xi) \partial_j \xi_p w_{lq} x_q + u_{il}(\xi) w_{lj} \Big) d\tau \right\| \leq \\ &\leq \max\{K_{pij}\}\delta R + \max\{K_{ij}\}, \quad \forall i,j \in \{1, \cdots, n\}. \end{aligned}$$

Thus, choosing

$$R = \frac{(2\delta + 1)\max\{K_{ij}\}}{1 - \max\{K_{pij}\}\delta},$$

$\mathcal{L}$ maps the ball B_R into itself. Since $\mathcal{L}$ is a continuous mapping, in order to conclude that $\mathcal{L}$ is compact it remains to show that $\mathcal{L}$ maps bounded sets into relatively compact sets. Indeed, for any $M \subset C^1(\bar{Q})$ bounded set and observing that C^1 is compactly imbedded in C, every sequence $\{z_m\} \subset \mathcal{L}(M)$ contains a convergent subsequence. Thus the Schauder fixed point theorem guarantees the existence of $z \in C^1(\bar{Q})$ such that $\mathcal{L}z = z$.

The derivative (2.12) is a consequence of $\mathcal{L}z = z$. ■

Now we are able to adapt the proof of the generalized Frobenius Theorem [15, pp. 167]. Note that the generalized Frobenius Theorem gives two equivalent statements requiring the existence of $z \in C^2$.

Theorem 2.1. *Suppose that the assumptions of Proposition 2.2 are fulfilled. If additionally the integrability condition holds*

$$\frac{\partial u_{ij}}{\partial \xi_p} u_{pq} w_{qm} w_{jl} = \frac{\partial u_{ij}}{\partial \xi_p} u_{pq} w_{ql} w_{jm} \tag{2.13}$$

then there exists $F \in C^1$ satisfying (1.5). Moreover, if $F \in C^2$ then u verifies (2.13).

Proof. Defining
$$v_{ij}(t) = \partial_j z_i(t,x) - t u_{il}(z(t,x)) w_{lj} \tag{2.14}$$
it satisfies the ordinary differential equation

$$v'_{ij}(t) = \partial_t \partial_j z_i(t,x) - u_{il}(z(t,x)) w_{lj} - t \frac{\partial u_{il}}{\partial z_p} \partial_t z_p w_{lj}. \tag{2.15}$$

From (2.12) we have

$$\partial_t \partial_j z_i = \frac{\partial u_{il}}{\partial z_p} \partial_j z_p w_{lq} x_q + u_{il} w_{lj}.$$

Introducing this relation and successively (2.11) of Proposition 2.2 in (2.15) we obtain

$$\begin{aligned} v'_{ij}(t) &= \frac{\partial u_{il}}{\partial z_p} \partial_j z_p w_{lq} x_q - t \frac{\partial u_{il}}{\partial z_p} \partial_t z_p w_{lj} \\ &= \frac{\partial u_{il}}{\partial z_p} \partial_j z_p w_{lq} x_q - t \frac{\partial u_{il}}{\partial z_p} u_{pm} w_{mq} x_q w_{lj}. \end{aligned}$$

Applying the assumption (2.13) it results the linear ODE

$$v'_{ij}(t) = \frac{\partial u_{il}}{\partial z_p} w_{lq} x_q v_{pj}. \tag{2.16}$$

Thus the initial condition $v_{ij}(0) = \partial_j z_i(0,x) = 0$ implies that the ODE (2.16) has the unique solution $v \equiv 0$. Setting $F(x) = z(1,x)$ and using (2.14) we get
$$\partial_j F_i = u_{il} w_{lj},$$
which concludes the proof of Theorem 2.1.

Finally, for $F \in C^2$, the condition (2.13) is due to the Schwartz property of functions of class C^2. ■

If we change u to the matrix valued map $u\,k$, then we realize the integrability condition (2.13) is in fact the one on the torsion stated in (1.4).

3 Invariant linear connections

Given a linear connection ∇ on a manifold M, one may define the subgroup $Diff(M,\nabla)$ of affine transformations of ∇. It is still a problem to find its dimension, as well as that of the orbit of ∇ under $Diff(M)$ in the space of linear connections.

One may also try to determine the linear connections on a manifold M which are invariant under a given set of diffeomorphisms. If we have a Lie group G, then it is easy to produce G-left invariant connections as bilinear maps $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of left invariant vector fields (cf. example 2, section 1.2).

Translation invariant connections in $\mathbb{R}^n$ are those for which Γ_{ij}^k are all constants. This is trivial to deduce from (1.2) applied to any map $F(x) = x + v$, with $v \in \mathbb{R}^n$.

A homothety invariant connection is one for which

$$\lambda\Gamma_{ij}^k(\lambda x) = \Gamma_{ij}^k(x) \tag{3.1}$$

as we may see taking $F(x) = \lambda x$ in the usual equation (1.5). If we want ∇ invariant for all λ, then ∇ is possibly very curved at the origin and certainly flat at infinity.

3.1 Over-determined systems of translation invariant connections in $\mathbb{R}^2$

Now we are going to find linear connections in $\mathbb{R}^2$ which are invariant for all translations plus one more single isomorphism $F(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)$. In view of the case of orbifolds, we are going to assume $\det F = \pm 1$ (we want the group generated by F to be finite).

The 8 equations from (1.2) are the following:

$$\begin{array}{lrcl}
i,j,k & & & \\
1,1,1 & a^2\Gamma_{11}^1 + ac\Gamma_{12}^1 + ac\Gamma_{21}^1 + c^2\Gamma_{22}^1 & = & a\Gamma_{11}^1 + b\Gamma_{11}^2 \\
1,1,2 & a^2\Gamma_{11}^2 + ac\Gamma_{12}^2 + ac\Gamma_{21}^2 + c^2\Gamma_{22}^2 & = & c\Gamma_{11}^1 + d\Gamma_{11}^2 \\
1,2,1 & ab\Gamma_{11}^1 + ad\Gamma_{12}^1 + cb\Gamma_{21}^1 + cd\Gamma_{22}^1 & = & a\Gamma_{12}^1 + b\Gamma_{12}^2 \\
1,2,2 & ab\Gamma_{11}^2 + ad\Gamma_{12}^2 + cb\Gamma_{21}^2 + cd\Gamma_{22}^2 & = & c\Gamma_{12}^1 + d\Gamma_{12}^2 \\
2,1,1 & ab\Gamma_{11}^1 + bc\Gamma_{12}^1 + ad\Gamma_{21}^1 + cd\Gamma_{22}^1 & = & a\Gamma_{21}^1 + b\Gamma_{21}^2 \\
2,1,2 & ab\Gamma_{11}^2 + bc\Gamma_{12}^2 + ad\Gamma_{21}^2 + cd\Gamma_{22}^2 & = & c\Gamma_{21}^1 + d\Gamma_{21}^2 \\
2,2,1 & b^2\Gamma_{11}^1 + bd\Gamma_{12}^1 + bd\Gamma_{21}^1 + d^2\Gamma_{22}^1 & = & a\Gamma_{22}^1 + b\Gamma_{22}^2 \\
2,2,2 & b^2\Gamma_{11}^2 + bd\Gamma_{12}^2 + bd\Gamma_{21}^2 + d^2\Gamma_{22}^2 & = & c\Gamma_{22}^1 + d\Gamma_{22}^2
\end{array} \tag{3.2}$$

One can reduce the system restricting to some particular subspace of linear connections. For instance, torsion free: then the system (3.2) reduces to 6 equations in 6 variables, because $\Gamma_{12}^k = \Gamma_{21}^k,\ k = 1, 2$. Indeed, (1.2) is symmetric in i, j if $\nabla = \mathrm{d} + \Gamma$ is torsion free.

Metric connections. A second case is that of metric connections with torsion (without torsion there is only the trivial, Levi-Civita connection):

$\Gamma^k_{ij} = -\Gamma^j_{ik}$, i.e. $\Gamma_i \in \mathfrak{so}$. Thus there are only two unknowns and the system (3.2) is given by

$$\begin{bmatrix} \Gamma^2_{11} & \Gamma^2_{21} \end{bmatrix} S = 0$$

where

$$S = \begin{bmatrix} b+ac & a^2-d & -ad+a & ab+c & bc & ab & -bd & b^2 \\ c^2 & ac & -cd & cb & cd+b & ad-d & a-d^2 & bd+c \end{bmatrix}.$$

In order to have $\operatorname{rk} S < 2$ we must have, e.g.,

$$\begin{cases} (b+ac)ac - (a^2-d)c^2 = c(ab+cd) = 0 \\ ad(d-1)(a-1) - abcd = ad(\pm 1 + 1 - a - d) = 0 \\ b^2(cd+b) - bc(bd+c) = b(b^2-c^2) = 0 \\ (a^2-d)cb - a^2bc - ac^2 = -c(bd+ac) = 0 \\ -bd(bd+c) - b^2(a-d^2) = -b(cd+ab) = 0 \end{cases}$$

Then we find non-trivial F given by

$$F_{a,\pm}(x,y) = \pm(x,-y), \quad \text{or} \quad F_{b,\pm}(x,y) = \pm(y,x).$$

Proposition 3.1. *$F_{a,\pm}$ and $F_{b,\pm}$ are the only non-trivial isomorphisms F of the plane for which there exist non-trivial metric, translation and F invariant connections.*

For $F_{a,+}$ the connections are given by $\Gamma^k_{ij} = 0$ for all i,j,k except $\Gamma^2_{21} = -\Gamma^1_{22}$.

For $F_{b,+}$ the connections are given by $\Gamma^k_{ij} = 0$, for all i,j,k except those satisfying also the condition $\Gamma^2_{11} = -\Gamma^1_{12} = -\Gamma^2_{21} = \Gamma^1_{22}$.

In both cases, $\nabla = \mathrm{d} + \Gamma$ is flat.

The proof follows from the system above and the curvature computations are trivial. Notice we may state corresponding results for the *minus* cases.

Symplectic connections. Another interesting type of conections is that of symplectic torsion free connections: Γ^k_{ij} is totally symmetric when contracted with the 2-form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ (see e.g. [2]), arising from the parallelism of ω, $\nabla\omega = 0$. This is the same as $\Gamma^1_{i1} = -\Gamma^2_{i2}$ or, equivalently, $\Gamma_i \in \mathfrak{sl}(2,\mathbb{R})$. In sum,

$$\Gamma^1_{11} = -\Gamma^2_{12} = -\Gamma^2_{21}, \qquad \Gamma^1_{21} = -\Gamma^2_{22} = \Gamma^1_{12}. \tag{3.3}$$

Now system (3.2) resumes to

$$\begin{bmatrix} a^2-a & 2ac & c^2 & -b \\ -c-2ac & -c^2 & 0 & a^2-d \\ ab+b & ad+bc-a & cd & 0 \\ -ad-cb+d & -cd-c & 0 & ab \\ ab+b & bc+ad-a & cd & 0 \\ -bc-ad+d & -cd-c & 0 & ab \\ b^2 & 2bd+b & d^2-a & 0 \\ -2bd & -d^2+d & -c & b^2 \end{bmatrix} \begin{bmatrix} \Gamma_{11}^1 \\ \Gamma_{12}^1 \\ \Gamma_{22}^1 \\ \Gamma_{11}^2 \end{bmatrix} = 0.$$

The rank of the essentially 6x4 matrix is less than 4 in situations our 'computer' does not obtain a pleasant result. But the case $a=-d=1,\ c=b=0$ is a solution. Then the connections are given by (3.3) and $\Gamma_{12}^1=\Gamma_{11}^2=0$. According to (1.3) we find

$$R_{121}^2 = -\Gamma_{21}^2\Gamma_{11}^1 = (\Gamma_{11}^1)^2.$$

Complex connections. We also have the case of complex or $\mathfrak{gl}(1,\mathbb{C})$-connections:

$$\Gamma_{11}^1=\Gamma_{12}^2, \quad \Gamma_{12}^1=-\Gamma_{11}^2, \quad \Gamma_{21}^1=\Gamma_{22}^2, \quad \Gamma_{22}^1=-\Gamma_{21}^2 \tag{3.4}$$

This gives an over-determined system as above, still unsolved according to its rank. If we moreover demand Γ torsion free, then the system reduces to 2 unknowns:

$$\begin{bmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 \end{bmatrix} S = 0$$

where

$$S = \left[\begin{matrix} a^2-c^2-a & 2ac-c & ab-cd-b \\ 2ac+b & c^2-a^2+d & ad+cb-a \end{matrix}\right. \\ \left.\begin{matrix} ad+bc-d & b^2-d^2+a & 2bd+c \\ -ab+cd-c & 2bd-b & -b^2+d^2-d \end{matrix}\right]$$

The condition for rk <2 remains to be deduced, but if we require $F \in GL(1,\mathbb{C})$, that is $F(x,y)=(ax+by,-bx+ay)$, then $F=\mathrm{Id}$ is the only isomorphism which admits that kind of invariant connections.

If we look for $\overline{F}$ of the previous kind, that is $F(x,y)=(ax+cy,cx-ay)$ then the equations for rk <2 resume to the vanishing of

$$\begin{aligned} (a^2-c^2-a)(c^2-a^2-a)-(2ac-c)(2ac+c) &= \\ &= -(a^2-c^2)^2+a^2-4a^2c^2+c^2 \\ &= (a^2+c^2)(1-a^2-c^2). \end{aligned}$$

Equivalently, $a^2+c^2=1$. Since we were hoping for $\det F=\pm1$ the result is automatic; thus we may write $a=\cos\theta,\ c=\sin\theta$, to find the condition

$$(\cos 2\theta-\cos\theta)\Gamma^1_{11}+(\sin 2\theta+\sin\theta)\Gamma^1_{12}=0.$$

In sum we have proved the following.

Proposition 3.2. *There exist complex, torsion free, translation and F invariant connections on $\mathbb{C}$ for each conjugate rotation $F(z)=e^{i\theta}\overline{z}$.*

Such connections are given by any $\lambda\in\mathbb{R}$ and

$$\begin{gathered}\Gamma^1_{11}=\Gamma^2_{12}=-\Gamma^1_{22}=\Gamma^2_{21}=-\lambda(\sin 2\theta+\sin\theta),\\ \Gamma^1_{12}=-\Gamma^2_{11}=\Gamma^1_{21}=\Gamma^2_{22}=\lambda(\cos 2\theta-\cos\theta).\end{gathered}$$

Moreover, these connections are flat.

The curvature is trivial since Γ is constant and a type $(1,0)$ form; since there are no type $(2,0)$ forms on the complex line, $R^\nabla=\mathrm{d}\Gamma+\Gamma\wedge\Gamma=0$.

In truth, all translation invariant complex connections in $\mathbb{C}$ are flat, cf. formulae (1.3,3.4).

3.2 Invariant connections on orbifolds $cL(p,q)$

Let ∇ be a holomorphic connection in $\mathbb{C}^n$ with coordinates $(z_1,\ldots,z_n)$, i.e. its Christoffel symbols for $\nabla_{\partial_{z_i}}\partial_{z_j}$ are holomorphic functions. Then the equations of an affine holomorphic transformation F are again determined by system (1.2) but with the x_j replaced by holomorphic coordinates $z_j=x_j+iy_j$. Indeed, since $F\cdot\partial_{\overline{z_j}}=0$, we must have

$$\nabla_{\partial_{z_j}}\partial_{\overline{z}_j}=\nabla_{\partial_{\overline{z}_i}}\partial_{z_j}=0$$

where as usual $\partial_{z_j}=\frac{1}{2}(\partial_{x_j}-i\partial_{y_j}),\ \partial_{\overline{z}_j}=\overline{\partial_{z_j}}$.

We recall the lens space $L(p,q)=S^3/\mathbb{Z}_p$, the orbit space for the action of $F(z_1,z_2)=(az_1,dz_2)$ on the 3-sphere, with $a,d\in\mathbb{C}$ such that $a^p=1,\ d=a^q$. In the study of the cone with a singularity

$$cL(p,q)=\{\lambda z:\ z\in L(p,q),\ \lambda\in\mathbb{R}^+\}=\mathbb{C}^2/\langle F\rangle,$$

there are invariant connections with meromorphic coefficients, as we shall see in the following example.

Example. $cL(p,q)$ admits a meromorphic connection with Christoffel symbols

$$\begin{array}{ll}\Gamma^1_{12}=\Gamma^1_{21}=\Gamma^2_{22}=\frac{1}{z_2} & \Gamma^1_{22}=\frac{z_1}{z_2^2}\\ \Gamma^2_{12}=\Gamma^1_{11}=\Gamma^2_{21}=\frac{1}{z_1} & \Gamma^2_{11}=\frac{z_2}{z_1^2}.\end{array}$$

The holomorphic curvatures of $\nabla = \mathrm{d} + \Gamma$ are

$$R^1_{121} = R^2_{122} = 0, \qquad R^1_{122} = \frac{2}{z_2^2}, \qquad R^2_{121} = -\frac{2}{z_1^2}.$$

For the proof, notice that, although ∇ is not translation invariant, we may still formally use system (3.2) viewing the Γ's composed with F on the left hand side. Then essentially two types of equation appear:

$$a\Gamma^1_{11} \circ F = \Gamma^1_{11}, \qquad a^2\Gamma^2_{11} \circ F = d\Gamma^2_{11},$$

and these equations have obvious solutions.

The search of meromorphic connections on orbifolds was studied in [12]; anyway our example seems to be original. The use of connections in this context has appeared in [7, 8]

We remark that the classification of orbifold singularities with complex structure is still an open problem and there are various approaches to it either through the Riemannian or the complex perspective — cf. [1, 4, 5] and the references therein to see the wealth of examples and geometries one might continue searching for.

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