On vector bundle manifolds with spherically symmetric metrics

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Key Words: vector bundle, metric connection, spherically symmetric metric, holonomy, G_2 manifold.
MSC 2010: 53C...

Metric on E and the Levi-Civita connection

Let (M, g_M) denote a Riemannian manifold.
Let $\pi : E \longrightarrow M$ be a rank-k vector bundle over M.
The vector bundle is endowed with a metric $g_E \in \Omega^0_M(S^2E^*)$ and a compatible **metric connection** D^E:

$$D^E g_E = 0.$$

The fibres $E_x = \pi^{-1}(x)$, $x \in M$ are smooth submanifolds, with tangent bundle the trivial bundle: $T(E_x) = E_x \times E_x$.
We have an exact sequence of vector bundles

$$0 \longrightarrow \mathcal{V} \longrightarrow TE \xrightarrow{\mathrm{d}\pi} \pi^* TM \longrightarrow 0$$

over the *manifold* E and the vertical bundle $\mathcal{V} \longrightarrow E$ identifies with $\pi^\star E \longrightarrow E$ (indeed, $\mathcal{V}_e = T_e(E_x) = \{e\} \times E_x = (\pi^\star E)_e$).

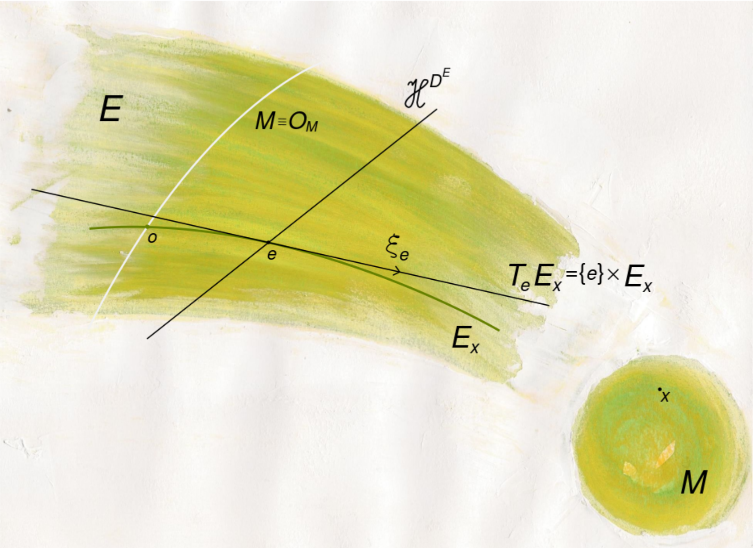
Next we use the connection D^E to induce a horizontal subspace and hence a splitting of TE. Since $\mathcal{H}^{D^E}$ is identified with the vector bundle π^*TM, through the restriction of the map $\mathrm{d}\pi$, we may finally write

$$TE = \mathcal{H}^{D^E} \oplus \mathcal{V} \simeq \pi^*TM \oplus \pi^\star E.$$

Any tangent vector $X = X^h + X^v$ at each point $e \in E$ has a well-defined decomposition.

We also have a natural vector field ξ, a **tautological section** of vertical directions, defined by $\xi_e = e \in \pi^\star E$. The important role played by ξ is shown through a projection onto $\mathcal{V}$ with kernel $\mathcal{H}^{D^E}$:

$$\pi^\star D^E_{X^h}\xi = 0 \qquad \pi^\star D^E_{X^v}\xi = X^v.$$


E
$M \equiv O_M$
$\mathscr{H}^{D^E}$
o
e
ξ_e
$T_e E_x = \{e\} \times E_x$
E_x
x
M

Indeed, if $s : M \to E$ is a section and $X = \mathrm{d}s_x(u) \in T_{s(x)}E$, then $\pi^\star D^E_{\mathrm{d}s(u)}\xi = \pi^\star(s^*\pi^\star D^E_u s^*\xi) = \pi^\star(D^E_u s)$.
In particular, we have $T(O_M) = \mathcal{H}^{D^E}_{|O_M}$.

We consider now the metrics g_M, g_E. Clearly the manifold E inherits a Riemannian structure $\pi^* g_M \oplus \pi^\star g_E$.
Letting ∇^M denote the Levi-Civita connection of M, the connection $D^{**} = \pi^*\nabla^M \oplus \pi^\star D^E$ **is a metric connection**, i.e. $D^{**}(\pi^* g_M \oplus \pi^\star g_E) = 0$. Its torsion satisfies

$$\begin{cases} \mathrm{d}\pi(T^{D^{**}}(X,Y)) = T^{\nabla^M}(\mathrm{d}\pi X, \mathrm{d}\pi Y) = 0 \\ (T^{D^{**}}(X,Y))^v = \pi^\star D^E_X Y^v - \pi^\star D^E_Y X^v - [X,Y]^v \; = \; \pi^\star R^E(X,Y)\xi \end{cases}$$

Recall that $\langle R^E(u,w) \;,\; \rangle_E$, $\forall u, w \in TM$ takes values in $\Lambda^2 E^*$.

We are interested in the following metric on the manifold E. First consider the function r defined by $r(e) = \langle e, e\rangle_E$ on E, i.e. the squared radial-distance to the 0 section. Since

$$r = \pi^\star g_E(\xi, \xi) \qquad \text{and} \qquad \pi^\star \overset{E}{D}_{X^v}\xi = X^v,$$

we have

$$\mathrm{d}r = 2(\pi^\star g_E)(\xi,\ ) = 2\langle \xi,\ \rangle_E = 2\xi^\flat.$$

The Riemannian structure on E we wish to study is defined by

$$\mathrm{g}_{M,E} = e^{2\varphi_1}\pi^* g_M \oplus e^{2\varphi_2}\pi^\star g_E$$

where φ_1, φ_2 are smooth scalar functions on E dependent only of r and smooth at $r = 0$ on the right

Remark. Next we use the notation $\varphi_1' = \frac{\partial \varphi_1}{\partial r}$.

We wish now to find a linear $\mathrm{g}_{M,E}$-metric connection over E keeping the same torsion of D^{**}. We must consider $\widetilde{D} = D^{**} + C$ with $C \in \Omega^0(S^2(T^*E) \otimes TE)$ given by (a, b, c_1, c_2 smooth functions of r)

$$\begin{aligned} C_X Y \ = \ & a\big(\xi^\flat(X)Y^h + \xi^\flat(Y)X^h\big) + \\ & + c_1\langle X, Y\rangle_M \xi + c_2\langle X, Y\rangle_E \xi + b\big(\xi^\flat(X)Y^v + \xi^\flat(Y)X^v\big)\ . \end{aligned}$$

Notice, for instance, $\langle X, Y\rangle_M$ stands for $\pi^* g_M(X^h, Y^h)$.

Theorem

The linear connection $\widetilde{D}$ on the Riemannian manifold E is a metric connection ($\widetilde{D}\,\mathrm{g}_{M,E} = 0$) if and only if

$$\begin{aligned} a &= 2\varphi_1' & \qquad c_1 &= -2\varphi_1' e^{2(\varphi_1-\varphi_2)} \\ b &= 2\varphi_2' & \qquad c_2 &= -2\varphi_2' \end{aligned} .$$

Since C is symmetric, we still have $T^{\widetilde{D}} = T^{D^{**}} = \pi^\star R^E(\ ,\ )\xi$. Let us abbreviate the notation for this $\mathcal{V}$-valued tensor: $\mathcal{R}^\xi = \pi^\star R^E(\ ,\ )\xi$.

The Levi-Civita connection $\nabla^{M,E}$ of the metric $\mathrm{g}_{M,E}$ is given by

$$\nabla^{M,E}_X Y = D^{**}_X Y + C_X Y + A_X Y - \frac{1}{2}\mathcal{R}^\xi(X,Y)$$

with C defined above and the $\pi^* TM$-valued 2-tensor A defined by

$$e^{2\varphi_1}\langle A(X,Y),Z\rangle_M = \frac{e^{2\varphi_2}}{2}\big(\langle\mathcal{R}^\xi(X,Z),Y\rangle_E + \langle\mathcal{R}^\xi(Y,Z),X\rangle_E\big).$$

Notice A is symmetric, so now we have $T^{\nabla^{M,E}} = 0$. Since $\widetilde{D} = D^{**} + C$ is a metric connection, we just have to verify

$$\mathrm{g}_{M,E}(A_X Y - \frac{1}{2}\mathcal{R}^\xi(X,Y),Z) = -\mathrm{g}_{M,E}(Y, A_X Z - \frac{1}{2}\mathcal{R}^\xi(X,Z)).$$

Parallel vector fields and isometries of $\mathrm{g}_{M,E}$

We try to find parallel vector fields, Killing v. f., isometries...

Let $x = (x^1, \ldots, x^m)$ be a chart of the base defined on an open subset $U \subset M$ ($\dim M = m$). If necessary, restricting to a smaller open subset we may take an **orthonormal frame** $\{e_1, \ldots, e_k\}$ **of** E **on** U. Hence we have a trivialization $\pi^{-1}(U) \simeq U \times \mathbb{R}^k$ with coordinates (x, y), linear on the fibres by assumption. Since any point $e \in \pi^{-1}(x)$ may be written as $e = \sum_\alpha y^\alpha e_\alpha$, the tautological vector field ξ satisfies $\xi_e = \sum_\alpha y^\alpha \pi^\star e_\alpha$. We have $r = \sum_\alpha (y^\alpha)^2$ and we denote $g_M(\partial_i, \partial_j) = g_{ij}$, where $\partial_i = \frac{\partial}{\partial x^i}$ are duals to the $\mathrm{d}x^j$. This has inverse matrix g^{jq}. We also let $\pi^*\partial_i$ denote the lift of ∂_i to the *horizontal* part of TE. The Christoffel symbols are defined by $\nabla^M_{\partial_i}\partial_j = \Gamma^{M,h}_{ij}\partial_h$ and $D^E_{\partial_i} e_\alpha = \Gamma^{E,\beta}_{i\alpha} e_\beta$. Throughout indices satisfy $1 \leq i, j, q, l \leq m$ and $1 \leq \alpha, \beta, \epsilon \leq k$, and Einstein summation convention is assumed. For the curvature tensor we denote $R^E_{\beta\alpha ij} = \langle R^E(\partial_i, \partial_j)e_\alpha, e_\beta \rangle_E$.

Note that $\partial_i = \partial_{i(x,y)}$ also makes sense in $\pi^{-1}(U)$, but such vector fields are not horizontal in general. It is easy to see that

$$\pi^*\partial_i = \partial_i - y^\alpha \Gamma^{E,\beta}_{i\alpha} \pi^\star e_\beta.$$

Notice $\pi^* g_M(\pi^*\partial_i, \pi^*\partial_j) = g_{ij}$ and $\pi^\star g_E(\pi^\star e_\alpha, \pi^\star e_\beta) = \delta^\beta_\alpha$. Hence

$$\mathrm{g}_{M,E}(\partial_i, \partial_j) = e^{2\varphi_1} g_{ij} + e^{2\varphi_2} y^\alpha y^\gamma \Gamma^{E,\beta}_{i\alpha} \Gamma^{E,\beta}_{j\gamma}.$$

Following the orthogonal decomposition of TE, any vector field on E is written as $Y = Y^j\pi^*\partial_j + B^\alpha\pi^\star e_\alpha$. Then we may develop four equations for $\nabla^{M,E}Y$ of different kind:

$$\begin{aligned}
(\nabla^{M,E}_{\pi^*\partial_i}Y)^q &= \frac{\partial Y^q}{\partial x^i} + Y^l\Gamma^{M,q}_{il} + ay^\alpha B^\alpha\delta^q_i + \frac{e^{2(\varphi_2-\varphi_1)}}{2}y^\alpha B^\beta R^E_{\beta\alpha ij}g^{jq} \\
(\nabla^{M,E}_{\pi^\star e_\beta}Y)^q &= \frac{\partial Y^q}{\partial y^\beta} + ay^\beta Y^q + \frac{e^{2(\varphi_2-\varphi_1)}}{2}y^\alpha Y^j R^E_{\beta\alpha jl}g^{lq} \\
(\nabla^{M,E}_{\pi^*\partial_i}Y)^\alpha &= \frac{\partial B^\alpha}{\partial x^i} + B^\beta\Gamma^{E,\alpha}_{i\beta} + c_1y^\alpha Y^j g_{ij} - \frac{1}{2}Y^j y^\beta R^E_{\alpha\beta ij} \\
(\nabla^{M,E}_{\pi^\star e_\beta}Y)^\alpha &= \frac{\partial B^\alpha}{\partial y^\beta} + c_2B^\beta y^\alpha + by^\beta B^\alpha + by^\epsilon B^\epsilon\delta^\beta_\alpha
\end{aligned}$$

Thus finding a (local) parallel vector field is non-trivial in general, even if we require Y to be horizontal or to be vertical.

Proposition

Assume **constant** *weights* $\varphi_1 = \varphi_1(r),\ \varphi_2 = \varphi_2(r)$.
(i) The only horizontal parallel vector fields Y on the manifold E are the horizontal lifts of parallel vector fields Y_0 of M for which $R^E(Y_0,\ ) = 0$.
(ii) The only vertical parallel vector fields on E are the vertical lifts of parallel sections of $\pi : E \longrightarrow M$.

Theorem

If the manifold E admits a $\nabla^{M,E}$-parallel non-vertical vector field, then M admits a ∇^M-parallel vector field. More precisely, every $\mathrm{g}_{M,E}$-parallel vector field over E restricts over O_M to an orthogonal sum of a parallel vector field of M and a parallel section of E.

For the more general equation of a Killing field $X \in \mathfrak{X}(E)$, i.e. a vector field such that $\mathcal{L}_X \mathrm{g}_{M,E} = 0$, equivalently, such that

$$\mathrm{g}_{M,E}(\nabla^{M,E}_Y X, Z) + \mathrm{g}_{M,E}(Y, \nabla^{M,E}_Z X) = 0, \quad \forall Y, Z \in \mathfrak{X}(E),$$

we cannot go much farther. We find

$$\begin{aligned}\mathcal{L}_X \mathrm{g}_{M,E} = e^{2\varphi_1}(\mathcal{L}_{X^h}\pi^* g_M)(Y,Z) + e^{2\varphi_2}(\mathcal{L}_{X^v}\pi^\star g_E)(Y,Z) + \\ +2ae^{2\varphi_1}\xi^\flat(X)\langle Y,Z\rangle_M + 2be^{2\varphi_2}\xi^\flat(X)\langle Y,Z\rangle_E + \\ +e^{2\varphi_2}\langle \mathcal{R}^\xi(X,Z),Y\rangle_E + e^{2\varphi_2}\langle \mathcal{R}^\xi(X,Y),Z\rangle_E.\end{aligned}$$

Infinitesimal isometries of the space E imply a complicated system. But we have the following quite immediate construction.

Suppose we have another Riemannian manifold M_1 and vector bundle $E_1 \longrightarrow M_1$ endowed with a metric structure g_{E_1} and metric connection D^{E_1}. Suppose also we have a parallel $(f^*D^{E_1} \circ \hat{f} = \hat{f} \circ D^{E})$ vector bundle **isometry** $\hat{f}$ along an **isometry** f of the base manifolds $(\pi_1 \circ \hat{f} = f \circ \pi)$:

$$\begin{array}{ccc} E & \xrightarrow{\hat{f}} & E_1 \\ \pi \downarrow & & \downarrow \pi_1 \\ M & \xrightarrow{f} & M_1. \end{array}$$

Theorem

In the above conditions, for the given same pair of functions φ_1, φ_2 on the radius of E and E_1, the map $\hat{f} : (E, \mathrm{g}_{M,E}) \longrightarrow (E_1, \mathrm{g}_{M_1,E_1})$ is an isometry.

Often one has an isometry $f : M \to M$ and one vector bundle $E \subset \otimes^p TM \bigotimes \otimes^q T^*M$, $p, q \in \mathbb{N}$, sub-vector bundle of the (p,q)-tensors on M, such that $f_*(E_x) = E_{f(x)}$, $\forall x \in M$.

Corollary

For any two functions of r, φ_1, φ_2, we have a 1-1 map

$$\mathrm{Isom}(M, g_M) \hookrightarrow \mathrm{Isom}(E, \mathrm{g}_{M,E}) \ .$$

We continue to deduce some basic properties of the metric.

Proposition

The Riemannian metric $\mathrm{g}_{M,E}$ and its Levi-Civita connection $\nabla^{M,E}$ satisfy the following properties:
(i) The zero section $O_M \subset E$ is totally geodesic.
(ii) The fibres of E are totally geodesic.
(iii) The vertical distribution $\mathcal{V} \subset TE$ is $\nabla^{M,E}$-parallel iff the horizontal distribution $\mathcal{H}^{D^E}$ is $\nabla^{M,E}$-parallel iff φ_1 is a constant and D^E is flat.

The integrability of the horizontal distribution is independent of the metric.

What other sections $s : M \to E$ embed M as a totally geodesic submanifold $s(M) = M^s$ of the Riemannian manifold $(E, \mathrm{g}_{M,E})$? It is easy to deduce

$$\mathrm{d}s(u) = u^h + D^E_u s \ \in T_{s_x} M^s \subset T_{s_x} E\,, \ \ \forall u \in T_x M.$$

Proposition

Let φ_1, φ_2 be **constants***.*
(i) Suppose that $R^E s = 0$. Then M^s is a totally geodesic submanifold of E if and only if

$$H^E(u,w)s := D^E_u D^E_w s - D^E_{\nabla^M_u w} s = 0\ , \quad \forall u, w \in TM.$$

(ii) Suppose that s_0 is a D^E-parallel section. Then the translation map $t : E \longrightarrow E$, $t(e) = e + s_0$, is an invariant map of $\nabla^{M,E}$.

Clearly, $H^E(u,w)s$ is *half* of $R^E_{u,w} s = H_{u,w} s - H_{w,u} s$. This generalized Hessian and its symmetric part are tensorial in u, w.

Geodesics

Recall the trivialization of E introduced earlier: $\pi^{-1}(U) \simeq U \times \mathbb{R}^k$ where U is the domain of a chart x of M. Also we use an orthonormal frame $\{e_\alpha\}_{\alpha=1,\dots,k}$, formed by sections of E on U.
A curve $\gamma = \gamma(t),\ t \in \mathbb{R}$, with image in $\pi^{-1}(U) \subset E$ may be written as a map:

$$\gamma = (\gamma^1, \dots, \gamma^m, y^1, \dots, y^k).$$

$\dot{\gamma}^i$ denotes derivative with respect to t.
Notice $\xi_\gamma = y^\alpha \pi^\star e_\alpha$.
In general, γ defines a section $y = y^\alpha e_\alpha$ of $E \longrightarrow M$ along $\pi \circ \gamma = (\gamma^1, \dots, \gamma^m)$.

Then along this same curve $\pi \circ \gamma$ we have

$$D^E_{\partial_t} y = \dot{y}^\beta e_\beta + \dot{\gamma}^i y^\alpha \Gamma^{E,\beta}_{i\alpha} e_\beta = (\dot{y}^\beta + \dot{\gamma}^i y^\alpha \Gamma^{E,\beta}_{i\alpha}) e_\beta = z^\beta e_\beta$$

where $z^\beta = \dot{y}^\beta + \dot{\gamma}^i y^\alpha \Gamma^{E,\beta}_{i\alpha}$.

Theorem

The curve γ in E is a geodesic of $\mathrm{g}_{M,E}$ if and only if we have

$$\begin{cases} \ddot{\gamma}^p + \dot{\gamma}^i \dot{\gamma}^j \Gamma^{M,p}_{ij} + 2a\dot{\gamma}^p z^\beta y^\beta + e^{2\varphi_2 - 2\varphi_1} \dot{\gamma}^i z^\beta y^\mu R^E_{\beta\mu i q} g^{qp} = 0 \\ \dot{z}^\alpha + \dot{\gamma}^i \dot{\gamma}^j c_1 g_{ij} y^\alpha + \dot{\gamma}^i z^\beta \Gamma^{E,\alpha}_{i\beta} - b z^\beta z^\beta y^\alpha + 2b z^\alpha z^\beta y^\beta = 0 \end{cases}$$

$\forall 1 \leq p \leq m,\ 1 \leq \alpha \leq k.$

A geodesic of $\mathrm{g}_{M,E}$ is a curve which satisfies $\gamma^*\nabla^{M,E}{}_{\partial_t}\dot\gamma = 0$, so first we deduce the canonical decomposition

$$\begin{aligned}\dot\gamma &= \dot\gamma^i\partial_i + \dot y^\beta\partial_{y^\beta}\\ &= \dot\gamma^i(\pi^*\partial_i + y^\alpha\Gamma^{E,\beta}_{i\alpha}\pi^\star e_\beta) + \dot y^\beta\pi^\star e_\beta\\ &= \dot\gamma^i\pi^*\partial_i + z^\beta\pi^\star e_\beta\end{aligned}$$

(notice this is essentially $\dot\gamma = \dot\gamma^i\pi^*\partial_i + \pi^\star(D^E_{\partial_t}y)$). Then

$$\gamma^*\nabla^{M,E}{}_{\partial_t}\dot\gamma = \ddot\gamma^i\pi^*\partial_i + \dot\gamma^i\nabla^{M,E}_{\dot\gamma}\pi^*\partial_i + \dot z^\beta\pi^\star e_\beta + z^\beta\nabla^{M,E}_{\dot\gamma}\pi^\star e_\beta\ ,$$

and since we have

$$\begin{aligned}\nabla^{M,E}_{\pi^\star e_\beta}\pi^*\partial_i &= ay^\beta\pi^*\partial_i + A_{\pi^\star e_\beta}\pi^*\partial_i\\ &= ay^\beta\pi^*\partial_i + \frac{e^{2\varphi_2-2\varphi_1}}{2}y^\mu R^E_{\beta\mu ij}g^{jq}\pi^*\partial_q\ ,\end{aligned}$$

we deduce the two summands

$$\begin{aligned}\dot\gamma^i\nabla^{M,E}_{\dot\gamma}\pi^*\partial_i &= \dot\gamma^i\dot\gamma^j\big(\Gamma^{M,l}_{ji}\pi^*\partial_l + c_1 g_{ij}y^\alpha\pi^\star e_\alpha - \frac{1}{2}\mathcal{R}^\xi(\pi^*\partial_j,\pi^*\partial_i)\big)\\ &\qquad + \dot\gamma^i z^\beta\nabla^{M,E}_{\pi^\star e_\beta}\pi^*\partial_i\\ &= \dot\gamma^i\dot\gamma^j\Gamma^{M,l}_{ij}\pi^*\partial_l + \dot\gamma^i\dot\gamma^j c_1 g_{ij}y^\mu\pi^\star e_\mu\\ &\qquad + \dot\gamma^i z^\beta(ay^\beta\pi^*\partial_i + \frac{e^{2\varphi_2-2\varphi_1}}{2}y^\mu R^E_{\beta\mu ij}g^{jq}\pi^*\partial_q)\end{aligned}$$

and

$$\begin{aligned}z^\beta\nabla^{M,E}_{\dot\gamma}\pi^\star e_\beta &= z^\beta\dot\gamma^j(\Gamma^{E,\mu}_{j\beta}\pi^\star e_\mu + ay^\beta\pi^*\partial_j + \frac{e^{2\varphi_2-2\varphi_1}}{2}y^\mu R^E_{\beta\mu jl}g^{lq}\pi^*\partial_q)\\ &\qquad + z^\beta z^\nu(c_2\delta^\beta_\nu y^\tau\pi^\star e_\tau + by^\nu\pi^\star e_\beta + by^\beta\pi^\star e_\nu)\ .\end{aligned}$$

Recalling $c_2 = -b$, adding and contracting, finishes the proof.

We recall that Γ^M, Γ^E and R^E depend only of the γ^i. Also the geodesics of M become geodesics of O_M, the zero section, as expected. Other lifts are quite 'singular'.

Proposition

Let γ be a curve in E which defines a non-vanishing parallel section y along the curve $\pi \circ \gamma$, thus having $\|y\|_E^2 = r_0 \neq 0$ a constant. Then γ is a geodesic of $\mathrm{g}_{M,E}$ if and only if $\pi \circ \gamma$ is a geodesic of M and $\varphi_1'(r_0) = 0$.

Proof.

This is immediate from above, since the assumption is $z^\alpha = 0,\ \forall 1 \leq \alpha \leq k$, and the term $\dot{\gamma}^i\dot{\gamma}^j c_1 g_{ij} = \|(\pi \dot{\circ} \gamma)\|_M^2 c_1$ (notice some $y^\alpha \neq 0$) varies only with $c_1(r)$ for any geodesic τ. ■

Regarding the completeness of the metric $\mathrm{g}_{M,E}$ we have the following observations. Recall the hypothesis that $\varphi_i,\ i=1,2$ are smooth at $r=0$ on the right. Then we conjecture that $\mathrm{g}_{M,E}$ **is complete if and only if the metric g_M on M is complete and also the totally geodesic fibres are complete**.
Our argument is first that the Riemannian metric is complete if and only if the induced metric space structure is complete, and that solutions for the above system exist on $U\times\mathbb{R}^k$. The completeness on the base and the bundle transition functions assure the smooth development up to infinity of geodesics contained in E.

Can't find a reference to this problem. Namely for Sasaki, $E=TM$.

But the argument seems to be ok, according to some applications due to Bryant and Salamon.

Spherically symmetric metrics on $\mathbb{R}^k$

A vertical geodesic of $\mathrm{g}_{M,E}$ is a geodesic which lies in the fibres of E. Any vertical geodesic is equivalent to a geodesic of E which is tangent to the fibres of E at just one point.

We may thus analyse these curves in the manifold $\mathbb{R}^k$ with metric $g_\varphi = e^{2\varphi(r)}((\mathrm{d}y^1)^2 + \cdots + (\mathrm{d}y^k)^2)$. In this case the usual $r = y^\alpha y^\alpha$.

This metric clearly has spherical symmetry (the canonical term).

We deduce the Levi-Civita connection even from the equations of $\nabla^{M,E}$. It is given by $\nabla_\beta \partial_\nu = -b\delta^\nu_\beta y^\mu \partial_\mu + by^\beta \partial_\nu + by^\nu \partial_\beta$.

From the above Theorem we deduce the geodesic equations:

$$\ddot{y}^\alpha + 2b\dot{y}^\alpha \dot{y}^\beta y^\beta - b\dot{y}^\beta \dot{y}^\beta y^\alpha = 0\ , \quad \forall 1 \leq \alpha \leq k\ .$$

One may find the Riemannian, Ricci and scalar curvatures.

The Riemannian curvature of $\mathrm{g}_{M,E}$

We start by the curvature of a simple case of the metric $\mathrm{g}_{M,E}$. We assume D^E is **flat**. Notice M may *have* curvature.
Let us take the connections $D^{**} = \pi^*\nabla^M \oplus \pi^\star D^E$ and $\widetilde{D} = D^{**} + C$ defined earlier, which now are both torsion free. So the L-C connection is $\nabla^{M,E} = \widetilde{D}$ On the way, we are assuming two weight functions φ_1, φ_2 of the squared-radius r. Next we use $\widetilde{R}$ for the Riemannian curvature tensor of $\mathrm{g}_{M,E}$. The following easy computations may be of some use:

$$\widetilde{R}(X^h,Y^h)Z^h = \pi^*R^M(X^h,Y^h)Z^h + 4r{\varphi_1'}^2 e^{2\varphi_1-2\varphi_2}(X^h\wedge Y^h)(Z^h)$$
$$\widetilde{R}(X^h,Y^h)Z^v = 0$$
$$\widetilde{R}(X^h,Y^v)Z^h = e^{2\varphi_1-2\varphi_2}\langle X^h,Z^h\rangle\big(4(\varphi_1''+{\varphi_1'}^2-2\varphi_1'\varphi_2')\xi^\flat(Y^v)\xi + 2(2r\varphi_1'\varphi_2'+\varphi_1')Y^v\big)$$
$$\widetilde{R}(X^h,Y^v)Z^v = \big(4(2\varphi_1'\varphi_2'-{\varphi_1'}^2-\varphi_1'')\xi^\flat(Y^v)\xi^\flat(Z^v) - 2(2r\varphi_1'\varphi_2'+\varphi_1')\langle Y^v,Z^v\rangle\big)X^h$$
$$\widetilde{R}(X^v,Y^v)Z^h = 0$$
$$\widetilde{R}(X^v,Y^v)Z^v = 4(\varphi_2''-{\varphi_2'}^2)\big(\xi^\flat(Z^v)(X^v\wedge Y^v)(\xi) - \langle X^v\wedge Y^v,\xi\wedge Z^v\rangle\xi\big) + 4(\varphi_2'+r{\varphi_2'}^2)(X^v\wedge Y^v)(Z^v).$$

We use $(u\wedge v)z = \langle u,z\rangle v - \langle v,z\rangle u$, so that constant curvature K that corresponds to $\widetilde{R}(u,v)z = -K(u\wedge v)z$.

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tangent to Q. By Proposition 7.2, Γ is reducible to a connection in Q. QED.

8. *Holonomy theorem*

We first prove the following result of Ambrose and Singer [1] by applying Theorem 7.1.

THEOREM 8.1. *Let $P(M, G)$ be a principal fibre bundle, where M is connected and paracompact. Let Γ be a connection in P, Ω the curvature form, $\Phi(u)$ the holonomy group with reference point $u \in P$ and $P(u)$ the holonomy bundle through u of Γ. Then the Lie algebra of $\Phi(u)$ is equal to the subspace of $\mathfrak{g}$, Lie algebra of G, spanned by all elements of the form $\Omega_v(X, Y)$, where $v \in P(u)$ and X and Y are arbitrary horizontal vectors at v.*

Figure: From "Kobayashi Nomizu"

The Riemannian curvature at the zero section

Let us again consider the connections D^{**} and $\widetilde{D} = D^{**} + C$. We show the computations of the curvature in general form. Let us denote

$$R^{\mathrm{g}_{M,E}} = R^{\nabla^{M,E}}.$$

Since $\xi = 0$ on O_M we have $C \wedge C\ _{|_o} = 0$ at any given point $o \in O_M$ of the zero section. It then follows by definition that

$$R^{\widetilde{D}}\ _{|_o} = \ R^{D^{**}} + \mathrm{d}^{D^{**}} C\ \ _{|_o}.$$

Recalling $\nabla^{M,E}$, the same reasons imply

$$\begin{aligned} R^{\mathrm{g}_{M,E}}\ _{|_o} &= R^{\widetilde{D}} + \mathrm{d}^{\widetilde{D}}(A - \frac{1}{2}\mathcal{R}^{\xi})\ \ _{|_o} \\ &= R^{D^{**}} + \mathrm{d}^{D^{**}} C + \mathrm{d}^{D^{**}}(A - \frac{1}{2}\mathcal{R}^{\xi})\ \ _{|_o}. \end{aligned}$$

Now, $X(\langle\xi,Y\rangle_E)=\langle X,Y\rangle_E+\langle\xi,\pi^\star D^E_X Y\rangle_E$ and hence, $\forall X,Y,Z,W\in TE$,

$$\begin{aligned}(D^{**}{}_X C_Y)Z\ \ _{|_o} &= D^{**}{}_X(C_Y Z)-C_Y(D^{**}{}_X Z)\ \ _{|_o}\\ &= a\langle X,Y\rangle_E Z^h+a\langle X,Z\rangle_E Y^h+c_1\langle Y,Z\rangle_M X^v+\\ &\qquad +c_2\langle Y,Z\rangle_E X^v+b\langle X,Y\rangle_E Z^v+b\langle X,Z\rangle_E Y^v.\end{aligned}$$

Here, $a=a_{|_0},\ b=b_{|_0}$, etc, just as for all other scalar functions — we recall, $c_1e^{2\varphi_2}=-ae^{2\varphi_1},\ a=2\varphi_1',\ b=2\varphi_2'=-c_2$. Then

$$\begin{aligned}&(\mathrm{d}^{D^{**}}C)(X,Y)Z\ \ _{|_o}\ =\\ &=\ (D^{**}{}_X C_Y)Z-(D^{**}{}_Y C_X)Z-C_{[X,Y]}Z\ \ _{|_o}\\ &=\ a\langle X,Z\rangle_E Y^h-a\langle Y,Z\rangle_E X^h+c_1\langle Y,Z\rangle_M X^v-c_1\langle X,Z\rangle_M Y^v\\ &\qquad +2b\langle X,Z\rangle_E Y^v-2b\langle Y,Z\rangle_E X^v.\end{aligned}$$

Since

$$\begin{aligned}\widetilde{D}_X(\mathcal{R}^\xi(Y,Z)) \;|_o &= \pi^\star D^E_X(\pi^\star R^E(Y,Z)\xi) \;|_o \\ &= (\pi^\star D^E_X \pi^\star R^E(Y,Z))\xi + \pi^\star R^E(Y,Z)\pi^\star D^E_X \xi \;|_o \\ &= R^E(Y,Z)X^v\end{aligned}$$

(with notation slightly abbreviated), we then have

$$(\widetilde{D}_X \mathcal{R}^\xi{}_Y)Z \;|_o = \widetilde{D}_X(\mathcal{R}^\xi(Y,Z)) - \mathcal{R}^\xi(Y,\widetilde{D}_X Z) \;|_o = R^E(Y,Z)X^v$$

and $g_{M,E}(\widetilde{D}_Y(A(X,Z)),W) \;|_o =$

$$\begin{aligned}&= Y\big(g_{M,E}(A(X,Z),W)\big) - g_{M,E}(A(X,Z),\widetilde{D}_Y W) \;|_o \\ &= \frac{1}{2} Y\big(e^{2\varphi_2}(\langle \mathcal{R}^\xi(X,W),Z\rangle_E + \langle \mathcal{R}^\xi(Z,W),X\rangle_E)\big) \;|_o \\ &= \frac{1}{2} e^{2\varphi_2}\big(\langle R^E(X,W)Y^v,Z\rangle_E + \langle R^E(Z,W)Y^v,X\rangle_E\big).\end{aligned}$$

Finally

$$\mathrm{g}_{M,E}(\mathrm{d}^{\widetilde{D}}(A-\frac{1}{2}\mathcal{R}^{\xi})(X,Y)Z,W)\ |_o\ =$$
$$=\mathrm{g}_{M,E}(\widetilde{D}_X(A-\frac{1}{2}\mathcal{R}^{\xi})_Y Z-\widetilde{D}_Y(A-\frac{1}{2}\mathcal{R}^{\xi})_X Z,W)\ |_o$$
$$=\mathrm{g}_{M,E}(\widetilde{D}_X((A-\frac{1}{2}\mathcal{R}^{\xi})(Y,Z))-\widetilde{D}_Y((A-\frac{1}{2}\mathcal{R}^{\xi})(X,Z)),W)\ |_o$$
$$=\frac{1}{2}e^{2\varphi_2}\big(\langle R^E(Y,W)X^v,Z\rangle_E+\langle R^E(Z,W)X^v,Y\rangle_E$$
$$-\langle R^E(Y,Z)X^v,W\rangle_E-\langle R^E(X,W)Y^v,Z\rangle_E$$
$$-\langle R^E(Z,W)Y^v,X\rangle_E+\langle R^E(X,Z)Y^v,W\rangle_E\big)\ .$$

Letting $R^{\mathrm{g}_{M,E}}(X,Y,Z,W)=\mathrm{g}_{M,E}(R^{\mathrm{g}_{M,E}}(X,Y)Z,W)$, we may deduce a set of formulas. First recall that

$$R^{D^{**}}=\pi^*R^M\oplus\pi^\star R^E\ .$$

Theorem

Let $x \in M$, $o \in O_M \subset E$ with $\pi(o) = x$. Then at point o

$$R_o^{\mathrm{g}_{M,E}}(X^h, Y^h, Z^h, W^h) = e^{2\varphi_1}\langle \pi^* R_x^M(X^h, Y^h)Z^h, W^h\rangle_M$$

$$R_o^{\mathrm{g}_{M,E}}(X^h, Y^h, Z^h, W^v) = 0$$

$$R_o^{\mathrm{g}_{M,E}}(X^h, Y^h, Z^v, W^v) = e^{2\varphi_2}\langle \pi^\star R_x^E(X^h, Y^h)Z^v, W^v\rangle_E$$

$$R_o^{\mathrm{g}_{M,E}}(X^h, Y^v, Z^h, W^v) = ae^{2\varphi_1}\langle X^h, Z^h\rangle_M \langle Y^v, W^v\rangle_E + \\ + \frac{1}{2}e^{2\varphi_2}\langle \pi^\star R_x^E(X^h, Z^h)Y^v, W^v\rangle_E$$

$$R_o^{\mathrm{g}_{M,E}}(X^v, Y^v, Z^h, W^h) = e^{2\varphi_2}\langle \pi^\star R_x^E(Z^h, W^h)X^v, Y^v\rangle_E$$

$$R_o^{\mathrm{g}_{M,E}}(X^v, Y^v, Z^v, W^h) = 0$$

$$R_o^{\mathrm{g}_{M,E}}(X^h, Y^v, Z^v, W^v) = 0$$

$$R_o^{\mathrm{g}_{M,E}}(X^v, Y^v, Z^v, W^v) = -2be^{2\varphi_2}\big(\langle X^v, W^v\rangle_E \langle Y^v, Z^v\rangle_E \\ -\langle X^v, Z^v\rangle_E \langle Y^v, W^v\rangle_E\big).$$

Recall E has rank k and M has dimension m.
In the following it is a remarkable surprise that the curvature of D^E has completely disappeared.

Theorem

The Ricci tensor $\mathrm{ric}^{\mathrm{g}_{M,E}}(X,Y) = \mathrm{tr}\, R^{\mathrm{g}_{M,E}}(\ ,X)Y$ *and the scalar curvature* $\mathrm{Scal}^{\mathrm{g}_{M,E}} = \mathrm{tr}_{\mathrm{g}_{M,E}} \mathrm{ric}^{\mathrm{g}_{M,E}}$ *satisfy (*$a = a_{|_0}$*,* $b = b_{|_0}$*, as well as with all other scalar functions):*

$$\mathrm{ric}_o^{\mathrm{g}_{M,E}}(X^h, W^h) = \mathrm{ric}_x^M(X^h, W^h) - ake^{2(\varphi_1-\varphi_2)}\langle X^h, W^h\rangle_M$$
$$\mathrm{ric}_o^{\mathrm{g}_{M,E}}(X^h, W^v) = 0$$
$$\mathrm{ric}_o^{\mathrm{g}_{M,E}}(X^v, W^v) = (2b(1-k) - am)\langle X, W\rangle_E$$

and also at o

$$\mathrm{Scal}_o^{\mathrm{g}_{M,E}} = e^{-2\varphi_1}\mathrm{Scal}_x^M + e^{-2\varphi_2}(2bk(1-k) - 2akm).$$

Corollary (“Einstein test”)

If the Riemannian manifold $(E, \mathrm{g}_{M,E})$ **is Einstein**, *hence satisfying* $\mathrm{ric}^{\mathrm{g}_{M,E}} = \lambda^E \mathrm{g}_{M,E}$, *then M is Einstein say with Einstein constant λ^M and at o we have*

$$\lambda^M e^{2\varphi_2 - 2\varphi_1} + a(m-k) + 2b(k-1) = 0.$$

Moreover

$$\begin{aligned}\lambda^E =& (2b(1-k) - am)e^{-2\varphi_2} \\ =& \lambda^M e^{-2\varphi_1} - ake^{-2\varphi_2}.\end{aligned}$$

The $R_o^{g_{M,E}}$ generate a Lie subalgebra of the orthogonal Lie algebra of T_oE. (Holonomy $\subset \mathfrak{o}(m+k) = \Lambda^2\mathbb{R}^{m+k}$.)
There are three types of operators $R_o^{g_{M,E}}(X,Y)$:

$$\begin{bmatrix} e^{2\varphi_1}R^M(X^h,Y^h) & 0 \\ 0 & e^{2\varphi_2}R^E(X^h,Y^h) \end{bmatrix}$$

$$\begin{bmatrix} 0 & -B(X^h,Y^v) \\ (B(X^h,Y^v))^\dagger & 0 \end{bmatrix},$$

$B(X^h,Y^v) = 2\varphi_1' e^{2\varphi_1}(X^h)^\flat \otimes (Y^v)^\flat + \frac{1}{2}e^{2\varphi_2}\langle R^E(X^h,\ )Y^v,\ \rangle_E,$
and

$$\begin{bmatrix} e^{2\varphi_2}\langle R^E(\ ,\ )X^v,Y^v\rangle_E & 0 \\ 0 & 4\varphi_2' e^{2\varphi_2}(X^v)^\flat \wedge (Y^v)^\flat \end{bmatrix}$$

($\cdot^\dagger$ is the adjoint with respect to the non-weighted metric).

By Ambrose-Singer these endomorphisms generate the local holonomy algebra.

The flat connection again

Suppose D^E is a flat connection on $E \longrightarrow M$ with the dimensions $m = \dim M$, $m+k = \dim E$, $k = \mathrm{rk}\, E$. At a point $o \in O_M$ we find the Riemannian curvature of $\mathrm{g}_{M,E}$ and find in the most general case, i.e. when both $\varphi_1'(0), \varphi_2'(0) \neq 0$, the three types of endomorphisms in $\mathfrak{o}(T_oE, \mathrm{g}_{M,E}) \simeq \Lambda^2\mathbb{R}^{m+k} = \Lambda^2\mathbb{R}^m \oplus \mathfrak{p} \oplus \Lambda^2\mathbb{R}^k$:

$$\begin{bmatrix} R^M & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -E_i^\alpha \\ (E_i^\alpha)^\dagger & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & e^\alpha \wedge e^\beta \end{bmatrix} \ \forall i, j, \alpha, \beta.$$

The matrices $E_i^\alpha = [\delta_i^p \delta_\alpha^\beta]_{p\beta}$ and those in the middle generate the subspace $\mathfrak{p}$ of dimension mk.

Since $[\mathfrak{p},\mathfrak{p}] = \mathfrak{o}(m) \oplus \mathfrak{o}(k)$, we find the first part of the following result.

Proposition

Let $\mathfrak{hol}^{\mathrm{g}_{M,E}}$ denote the whole holonomy Lie algebra.
(i) If $\varphi_1'(0) \neq 0$, then $\mathfrak{hol}^{\mathrm{g}_{M,E}} = \mathfrak{o}(m+k)$.
(ii) If $\varphi_1'(0) = 0 \neq \varphi_2'(0)$, then $\mathfrak{hol}^{\mathrm{g}_{M,E}} \supseteq \mathfrak{hol}^M \oplus \mathfrak{o}(k)$.
(iii) $\mathfrak{hol}^{\mathrm{g}_{M,E}} \supseteq \mathfrak{hol}^M$, with equality if both φ_1, φ_2 are constant.

Hermitian tangent bundle with generalized Sasaki metric

Given any (Riemannian) manifold M, the generalized Sasaki almost Hermitian structure consists of the $\mathrm{g}_{M,E}$**-compatible almost complex structure** J^ψ **on the manifold** $E = TM$ defined by

$$J^\psi = e^{-\psi}B - e^{\psi}B^\dagger$$

where $\psi = \varphi_2 - \varphi_1$ and the well-defined endomorphism $B : TTM \longrightarrow TTM = \pi^*TM \oplus \pi^\star TM : \ BZ^h = Z^v, \ BZ^v = 0.$ (B cannot be defined on other vector bundles; it is the **diagonal group structure** $\Delta \subset \mathrm{GL}(m) \times \mathrm{GL(m)}$). Notice:

$$J^\psi \circ J^\psi = (e^{-\psi}B - e^{\psi}B^\dagger) \circ (e^{-\psi}B - e^{\psi}B^\dagger) = -1_{TTM}.$$

J^ψ generalizes the case $\varphi_1 = \varphi_2 = 0$, due to Sasaki.

Indeed we have an almost Hermitian structure:

$$\mathrm{g}_{M,E}(J^{\psi}\ \ ,J^{\psi}\ \ ) = \mathrm{g}_{M,E}(\ \ ,\ \ ).$$

It follows that the associated symplectic 2-form $\omega^{\overline{\psi}} := J^{\psi} \lrcorner \mathrm{g}_{M,E}$ satisfies (we let $\overline{\psi} = \varphi_2 + \varphi_1$)

$$\omega^{\overline{\psi}} = e^{\overline{\psi}}\omega^0.$$

Proposition

For $\dim M > 1$*, the 2-form* $\omega^{\overline{\psi}}$ *on* $E = TM$ *is symplectic if and only if* $\overline{\psi}$ *is a constant.*

Proof.

First $\mathrm{d}(e^{\overline{\psi}}\omega^0) = \mathrm{d}e^{\overline{\psi}} \wedge \omega^0 + e^{\overline{\psi}}\mathrm{d}\omega^0$. Since $D^{**}\omega^0 = 0$ we find $\mathrm{d}\omega^0(X,Y,Z) = \omega^0(T^{D^{**}}(X,Y),Z) + \omega^0(T^{D^{**}}(Y,Z),X) + \omega^0(T^{D^{**}}(Z,X),Y)$. Since $T^{D^{**}}(\ ,\ ) = \pi^\star R^E(\ ,\ )\xi$, we always have $\mathrm{d}\omega^0 = 0$ by Bianchi identity. ∎

Now, in arXiv1609.03125, we have deduced when J^ψ is **integrable**. We found non trivial solutions — however, a **particular case of a result of V. Oproiu and N. Papaghiuc**, cf. arXiv1609.03125. In any dimension, the unique non-flat solutions are the tangent **disk bundles** $D_{r_0}M = \{u \in TM : \ \|u\|^2 < r_0\}$ of a real base manifold M of constant sectional curvature $\kappa \neq 0$ with any squared-radius $r_0 \in \mathbb{R}^+$ and metric satisfying $e^{\varphi_1 - \varphi_2} = \sqrt{1+\kappa r}$. For a complete metric, $M = S^m$ is a sphere $(\kappa > 0)$ and $r_0 = +\infty$. Such disk bundles are also Kählerian if we take $\overline{\psi} = 0$ (i.e. $\varphi_1 = -\varphi_2$). The metric is given by

$$\mathrm{g}_{M,DM} = \sqrt{1+\kappa r}\,\pi^* g_M \oplus \frac{1}{\sqrt{1+\kappa r}}\,\pi^\star g_{TM}$$

where $r_0 = -1/\kappa$ if $\kappa < 0$. And $r_0 = +\infty$ otherwise.

By analogy with Bryant and Salamon, the metric is **complete** if and only if the rays to infinity have infinite length:

$$\int_0^{r_0} \left\| \frac{\mathrm{d}}{\mathrm{d}t} \right\|_{M,DM} \mathrm{d}t = \int_0^{r_0} \frac{1}{\sqrt[4]{1+\kappa t^2}} \mathrm{d}t.$$

The holonomy lies in the unitary Lie algebra $\mathfrak{u}(m)$.
At the zero section we find

$$\mathfrak{u}(m) \text{ for } m > 2 \qquad \text{and} \qquad \mathfrak{su}(2) \text{ for } m = 2.$$

We find the possible Einstein constant:

$$\Lambda_{DM} = \frac{1}{2}(m-2)\kappa.$$

The zero section O_M can only tell us about the whole geometry of E when we have J^{ψ} parallel.

Metrics with G_2 holonomy

G_2 manifolds are very *active* in String theory...
$\mathrm{G}_2 := \mathrm{Aut}\,\mathbb{O} \subset \mathrm{SO}(7)$ is a simplyconnected, compact, simple Lie group of dimension 14.

A 7-dimensional manifold $\mathcal{E}$ carries a G_2 structure if it admits a certain stable 3-form ϕ. This 3-form yields a metric g_ϕ such that ϕ becomes $\phi(X,Y,Z) = \langle X \cdot Y, Z\rangle_\phi$ and each 7-dim $T_e\mathcal{E}$ inherits the structure of $\Im(\mathbb{O})$.

Theorem (Fernández-Gray)

$\nabla^{g_\phi}\phi = 0$ *iff* $\mathrm{d}\phi = \mathrm{d} * \phi = 0.$

Famous example by R. Bryant and S. Salamon on $\Lambda^2_- T^*M \longrightarrow M$ produces true complete holonomy G_2.

In arXiv1401.7314 we study some generalizations of the metrics of Bryant-Salamon on the vector bundle

$$E := \Lambda^2_- T^*M \longrightarrow M$$

of self-dual and anti-self-dual 2-forms.
M denotes an **oriented Riemannian 4-manifold**.
The constructed G_2 structures on E are parallel for positive scalar curvature self-dual Einstein manifolds, this is, S^4 and $\mathbb{CP}^2$.

A change of orientation is ok, but in working with Λ^2_+ one finds a mirror construction for **negative** scalar curvature and finds an **unknown** number of new examples of Riemannian 7-manifolds with G_2 holonomy. In particular for the Einstein base $M = \mathcal{H}^4$ and the anti-self-dual $\mathcal{H}^2_{\mathbb{C}}$, respectively, the real and complex hyperbolic spaces.

When trying to find the holonomy subgroup of G_2, the Lie theory for those *new* symmetric spaces fails. The arguments of Bryant-Salamon cannot be reproduced for $\mathrm{Scal}^M \leq 0$.
Now, our theory gives a general proof that the holonomy groups are the whole G_2. (The study includes the $\mathrm{Scal}^M = 0$ base, which yields a different conclusion.)

Remark. Similar metrics on vector bundle manifolds with G_2 holonomy were also found by G. W. Gibbons, D. N. Page and C. N. Pope, also with the bundles over S^4 and $\mathbb{CP}^2$.
They too do not see the $\mathrm{Scal}^M \leq 0$ cases.

We have a manifold $E=\Lambda^2_\pm T^*M$ and a G_2 metric $\mathrm{g}_{M,E}$ in general. We shall not be focused on the G_2 structure, the 3-form ϕ, which determines the metric.

Given an oriented orthonormal frame $\{e^4,e^5,e^6,e^7\}$ of T^*M on an open subset, we have a frame on E on the same open subset defined by

$$e^1=e^{45}\pm e^{67}\ ,\qquad e^2=e^{46}\mp e^{57}\ ,\qquad e^3=e^{47}\pm e^{56}.$$

The metric g_E on $E=E_\pm=\Lambda^2_\pm T^*M\rightarrow M$, as a vector bundle, is such that $\{e^1,e^2,e^3\}$ is an orthonormal frame.

This implies e.g. $r=\|e^1\|_E=\frac{1}{2}e^1(e_1)$.

We consider:
• Λ^2_- when the base manifold M is Einstein and self-dual (i.e. has vanishing anti-self dual Weyl tensor $W_- = 0$).
• Λ^2_+ when M is Einstein anti-self-dual ($W_+ = 0$).
The vector bundles $E_\pm$ inherit a metric connection from the Levi-Civita connection ∇^M (this commutes with $*$ operator).

It is easy to see the "Bryant-Salamon" metric on $E_\pm$ is a spherically symmetric metric $\mathrm{g}_{M,E}$ with certain weight functions:

$$\mathrm{g}_{M,E} = \sqrt{2\tilde{c}_0^2 sr + \tilde{c}_1}\,\pi^* g_M \oplus \frac{\tilde{c}_0^2}{\sqrt{2\tilde{c}_0^2 sr + \tilde{c}_1}}\,\pi^\star g_E$$

where $\tilde{c}_0, \tilde{c}_1 > 0$ are constants and $s = \frac{1}{12}\mathrm{Scal}^M$.
We keep the two constants $\tilde{c}_0, \tilde{c}_1$, although we may further normalize...

We already know the metric $\mathrm{g}_{M,E}$ has holonomy inside G_2 (i.e. ϕ is parallel for the Levi-Civita connection itself induces).
Again, for $\mathrm{Scal}^M < 0$ we are forced to restrict the study to the open disk bundle of radius $\sqrt{r_0}$, where $r_0 = -\tilde{c}_1/2s$,

$$D_{r_0,\pm}M = \{e \in E : \ \|e\|_E^2 < r_0\}.$$

The spherically symmetric metric weights are given by

$$\varphi_1(r) = \frac{1}{4}\log(2\tilde{c}_0^2 sr + \tilde{c}_1) \qquad \varphi_2(r) = -\frac{1}{4}\log(2\tilde{c}_0^2 sr + \tilde{c}_1) + \log\tilde{c}_0.$$

Then

$$e^{2\varphi_1(0)} = \tilde{c}_1^{\frac{1}{2}} \qquad e^{2\varphi_2(0)} = \tilde{c}_0^2\tilde{c}_1^{-\frac{1}{2}}$$

$$\varphi_1'(r) = \frac{\tilde{c}_0^2 s}{2(2\tilde{c}_0^2 sr + \tilde{c}_1)} \qquad \varphi_2'(r) = -\frac{\tilde{c}_0^2 s}{2(2\tilde{c}_0^2 sr + \tilde{c}_1)}$$

and

$$\varphi_1'(0) = \frac{\tilde{c}_0^2 s}{2\tilde{c}_1} = -\varphi_2'(0).$$

Regarding the famous coefficients of the map "C", defined by $a = 2\varphi_1',\ b = 2\varphi_2' = -c_2,\ c_1 = -2\varphi_1' e^{2\varphi_1 - 2\varphi_2}$, we find at 0

$$a = -b = c_2 = \frac{\tilde{c}_0^2 s}{\tilde{c}_1} \qquad c_1 = -s.$$

Any holonomy G_2 manifold is Ricci flat (E. Bonan). In our case this may be **confirmed by the "Einstein test"**.
Indeed, we have both

$$2b(1-k) - am = -4(b+a) = 0$$

and

$$\lambda^{M} e^{-2\varphi_1} - ake^{-2\varphi_2} = 3s\tilde{c}_1^{-\frac{1}{2}} - 3\frac{\tilde{c}_0^2 s}{\tilde{c}_1}\tilde{c}_0^{-2}\tilde{c}_1^{\frac{1}{2}} = 0.$$

We now write our main result for the metrics $\mathrm{g}_{M,E}$ on $\Lambda^2_- T^*M$ if $s>0$ and $D_{r_0,+}M$ if $s<0$.

Theorem

For $s\neq 0$, the holonomy group of $\mathrm{g}_{M,E}$ is the Lie group G_2.

For the proof...

Recall the decomposition of the curvature tensor of 4-manifolds under the Lie algebra $\mathfrak{o}(4)=\mathfrak{o}(3)\oplus\mathfrak{o}(3)$. The symmetric operator on 2-forms defined by

$$\langle\mathcal{R}(e_\alpha\wedge e_\beta),e_\gamma\wedge e_\delta\rangle_M=-\langle R^M(e_\alpha,e_\beta)e_\gamma,e_\delta\rangle_M=R^M_{\alpha\beta\gamma\delta}$$

gives rise to an irreducible decomposition respecting $\Lambda^2_+\oplus\Lambda^2_-$

$$\mathcal{R}=\begin{bmatrix} W_++s1_3 & \mathrm{ric}_0 \\ {\mathrm{ric}_0}^\dagger & W_-+s1_3\end{bmatrix}.$$

$W = W_+ + W_-$ is the Weyl tensor, traceless. The map ric_0 is the traceless part of ric^{g_M}. It follows $s = \frac{1}{12}\mathrm{Scal}^M$.
Now the curvature of the vector bundle E is given in the frame (e^1, e^2, e^3) by

$$R^E(e^1, e^2, e^3) = (e^1, e^2, e^3)\rho \qquad \text{where} \qquad \rho = \begin{bmatrix} 0 & -\rho^3 & \rho^2 \\ \rho^3 & 0 & -\rho^1 \\ -\rho^2 & \rho^1 & 0 \end{bmatrix}.$$

In other words

$$R^E e^i = \rho^k e^j - \rho^j e^k, \qquad \forall \text{ cycle } (ijk) = (123).$$

We may write again ρ, more precisely each ρ^i_+ and ρ^i_-, as a linear combination of the self-dual and anti-self dual 2-forms.

Taking a dual frame of the e^4, e^5, e^6, e^7, we find respective 2-vectors e_1, e_2, e_3, which verify $e^i_\pm(e_{\pm,j}) = 2\delta^i_j$, $e^i_\pm(e_{\mp,j}) = 0$, $\forall i, j = 1, 2, 3$. Careful computations yield:

$$\rho^i_+(e_{+,j}) = -\mathcal{R}_{ij} \qquad \rho^i_\pm(e_{\mp,j}) = \mp\mathcal{R}_{i\bar{j}} \qquad \rho^i_-(e_{-,j}) = +\mathcal{R}_{\bar{i}\bar{j}}.$$

Notice, for instance, $\mathcal{R}_{ij} = \langle \mathcal{R}e_i, e_j \rangle_M$.
If M is Einstein, equivalently, if $\mathrm{ric}_0 = 0$, then s is a constant.

M is self-dual if $W = W_+$. Self-dual and Einstein is the same as

$$W_- = 0 \quad \Longleftrightarrow \quad \rho^i_- = se^i_- \ , \ \forall i = 1, 2, 3.$$

Anti-self-duality corresponds to $W = W_-$. Together with the Einstein condition, that implies $\rho^i_+ = -se^i_+$.
All together, the hypothesis is that $\rho^i = \mp se^i$.

Now, we just have to prove $\dim \mathfrak{hol}^{\mathrm{g}_{M,E}} = 14 = \dim \mathrm{G}_2$.
By Ambrose-Singer, the holonomy is generated by those matrices in $\mathfrak{o}(7)$ found earlier:

$$R^{\mathrm{g}_{M,E}}(X^h, Y^h) = \begin{bmatrix} \tilde{c}_1^{\frac{1}{2}} R^M(X^h, Y^h) & 0 \\ 0 & \tilde{c}_0^2 \tilde{c}_1^{-\frac{1}{2}} R^E(X^h, Y^h) \end{bmatrix}$$

$$R^{\mathrm{g}_{M,E}}(X^h, Y^v) = \begin{bmatrix} 0 & -B \\ B^\dagger & 0 \end{bmatrix}$$

with $B(X^h, Y^v) = a\tilde{c}_1^{\frac{1}{2}}(X^h)^\flat \otimes (Y^v)^\flat + \frac{1}{2}\tilde{c}_0^2\tilde{c}_1^{-\frac{1}{2}}\langle R^E(X^h, \ )Y^v, \ \rangle_E$
and

$$R^{\mathrm{g}_{M,E}}(X^v, Y^v) = \begin{bmatrix} \tilde{c}_0^2 \tilde{c}_1^{-\frac{1}{2}} \langle R^E(\ ,\ )X^v, Y^v\rangle_E & 0 \\ 0 & 2b\tilde{c}_0^2\tilde{c}_1^{-\frac{1}{2}}(X^v)^\flat \wedge (Y^v)^\flat \end{bmatrix}$$

Notice we may consider the horizontal lift of e_i as well as the vertical lift of e^i, which we have denoted by $\pi^\star e^i$. Recall $\langle e^i, e^j\rangle_E = \frac{1}{2}\langle e^i, e^j\rangle_M = \delta_{ij},\ \forall i,j = 1,2,3$. We then conclude various identities on the manifold E. First, in coherence with the above,

$$\frac{1}{2}\langle R^M(e_k), e_i\rangle_M = -\frac{1}{2}\mathcal{R}_{ki} = \pm\frac{1}{2}\rho^k_\pm(e_i) = -s\delta_{ik}$$

and hence $R^M(e_k) = -se^k$ (the horizontal lift, the pullback). Second, in positive order (ijk),

$$\langle R^E(e_k)\pi^\star e^i, \pi^\star e^j\rangle_E = \rho^k(e_k) = \mp 2s = \mp 2s(\pi^\star e^i \wedge \pi^\star e^j)(e_i, e_j).$$

Finally, the orthogonal maps $R^{\mathrm{g}_{M,E}}(e^h_k)$ and $R^{\mathrm{g}_{M,E}}(\pi^\star e^i, \pi^\star e^j)$,

$$R^{\mathrm{g}_{M,E}}(e^h_k) = \begin{bmatrix} -\tilde{c}_1^{\frac{1}{2}} s e^k & 0 \\ 0 & \mp 2s\tilde{c}_0^2 \tilde{c}_1^{-\frac{1}{2}} \pi^\star e^i \wedge \pi^\star e^j \end{bmatrix},$$

$$R^{\mathrm{g}_{M,E}}(\pi^\star e^i, \pi^\star e^j) = \begin{bmatrix} \mp\tilde{c}_0^2 \tilde{c}_1^{-\frac{1}{2}} s e^k & 0 \\ 0 & -2s\tilde{c}_0^4 \tilde{c}_1^{-\frac{3}{2}} \pi^\star e^i \wedge \pi^\star e^j \end{bmatrix},$$

are *the same*:

$$\pm\frac{\tilde{c}_0^2}{\tilde{c}_1} R^{\mathrm{g}_{M,E}}(e^h_k) = R^{\mathrm{g}_{M,E}}(\pi^\star e^i, \pi^\star e^j)$$

and we have proved all these 6 maps generate a 3-dimensional subspace.

There is another 3-dimensional subspace of maps, non-vanishing just in the 4×4-square, generated by the

$$R^{\mathrm{g}_{M,E}}(e^h_k) = \begin{bmatrix} \tilde{c}_1^{\frac{1}{2}} R^M(X^h, Y^h) & 0 \\ 0 & 0 \end{bmatrix}.$$

They refer to $W_{\mp} + s1_3$ and do not vanish because $s \neq 0$. (*Sometimes* W_+ or W_- do not vanish either.)

We are left to prove the $R^{\mathrm{g}_{M,E}}(X^h,Y^v)$ generate an 8-dimensional subspace. Let us take any $\alpha=4,5,6,7$. Letting $\theta^i=\langle\pi^\star e^i,\ \rangle_E$, we have:

$$
\begin{aligned}
&R^{\mathrm{g}_{M,E}}(e_\alpha,\pi^\star e^i,Z^h,W^v)=\\
&=a\tilde{c}_1^{\frac12}\langle e_\alpha,Z^h\rangle_M\langle\pi^\star e^i,W^v\rangle_E+\frac12\tilde{c}_0^2\tilde{c}_1^{-\frac12}\langle R^E(e_\alpha,Z^h)\pi^\star e^i,W^v\rangle_E\\
&=\frac{\tilde{c}_0^2 s}{2\tilde{c}_1^{\frac12}}\big(2e^\alpha(Z^h)\langle\pi^\star e^i,W^v\rangle_E\mp e^k(e_\alpha,Z^h)\langle\pi^\star e^j,W^v\rangle_E\pm\\
&\qquad\qquad e^j(e_\alpha,Z^h)\langle\pi^\star e^k,W^v\rangle_E\big).
\end{aligned}
$$

Hence

$$
R^{\mathrm{g}_{M,E}}(e_\alpha,\pi^\star e^i)=\frac{\tilde{c}_0^2 s}{2\tilde{c}_1^{\frac12}}\big(2e^\alpha\wedge\theta^i\mp e_\alpha\lrcorner e^k\wedge\theta^j\pm e_\alpha\lrcorner e^j\wedge\theta^k\big).
$$

Computing case by case we get **four linearly independent** families of three *similar* 2-forms. Writting $V_{\alpha i} = e^{\alpha} \wedge \theta^{i}$, we get for instance

$$\begin{cases} R^{\mathrm{g}_{M,E}}(e_4, \pi^\star e^1) = 2V_{41} \mp V_{72} \pm V_{63} \\ R^{\mathrm{g}_{M,E}}(e_7, \pi^\star e^2) = 2V_{72} \mp V_{41} + V_{63} \\ R^{\mathrm{g}_{M,E}}(e_6, \pi^\star e^3) = 2V_{63} \pm V_{41} + V_{72} \end{cases} .$$

These forms are linearly dependent. In fact, each and all the following matrices, corresponding to the four families, have rank 2:

$$\begin{bmatrix} 2 & \mp 1 & \pm 1 \\ \mp 1 & 2 & 1 \\ \pm 1 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & \mp 1 \\ 1 & 2 & \pm 1 \\ \mp 1 & \pm 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & \pm 1 & 1 \\ \pm 1 & 2 & \mp 1 \\ 1 & \mp 1 & 2 \end{bmatrix}$$

and therefore the curvature generates a subspace of dimension 8, q.e.d.

The case $s = 0$ implies constant φ_1, φ_2.
Recall the famous K3 surfaces are Kähler surfaces (4 real dim); with the Calabi-Yau metric, they have a fixed orientation, are non-flat, Ricci-flat and anti-self-dual.
Anti-self-duality occurs necessarily with every scalar flat Kähler surface (A. Derdzinski).
K3 surfaces have thus holonomy $\mathrm{SU}(2)$.
K3 surfaces and quotients of the 4-torus by finite groups give us all the compact spin Ricci-flat Kähler surfaces and hence anti-self-dual 4-manifolds (C. Lebrun).
The flat torus case being trivial, we proceed.

Theorem

For any K3 surface M, the G_2 metrics on $E_+ = \Lambda^2_+ T^*M$ have holonomy $\mathrm{SU}(2) \subset \mathrm{G}_2$.

Proof.

Apply global formulas, since E is flat for $s = 0$ as we have seen. ■

Resuming with $\mathrm{Scal}^M \neq 0$. Let us stress we are now completely sure the spaces

$$D_{r_0,\pm}\mathcal{H}^4 \qquad \text{and} \qquad D_{r_0,+}\mathcal{H}^2_{\mathbb{C}} \ , \tag{1}$$

with the metric $\mathrm{g}_{M,E}$, have G_2 holonomy.

Let us see a topological proof for the Bryant-Salamon metrics. This third independent proof is, again, suitable only for the positive Scal^M cases.

Proposition

The G_2 metric on $\Lambda^2_- T^*S^4$ has holonomy equal to G_2.

Proof.

A theorem of Bryant assures that if the holonomy group is contained in G_2 and the metric does not admit parallel vector fields, then the subgroup coincides with the whole group. Now if E_- had a parallel vector field $Y = Y^h + Y^v$ for the G_2 metric, then this would restrict over the zero section O_M to the sum of a parallel vector field Y^h and a parallel section Y^v. These fields would have constant norm. But S^4 does not have non-vanishing vector fields, nor it admits a non-degenerate 2-form field (an almost-complex structure). Of course every self- or anti-self-dual 2-form is non-degenerate. ∎

Analogously, for $\mathbb{CP}^2$; because it does not admit a non-vanishing vector field, nor a Kähler structure compatible with the Fubini-Study metric and inducing the reversed orientation. It is well-known that $\overline{\mathbb{CP}}^2$ is not even a complex manifold.

Ann Glob Anal Geom (2017) 51:129–154
DOI 10.1007/s10455-016-9528-y

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On vector bundle manifolds with spherically symmetric metrics

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Received: 11 February 2016 / Accepted: 15 September 2016 / Published online: 26 September 2016

Abstract We give a general description of the construction of weighted spherically symmetric metrics on vector bundle manifolds, i.e. the total space of a vector bundle $E \longrightarrow M$, over a Riemannian manifold M, when E is endowed with a metric connection. The tangent