Twistorial constructions of special Riemannian manifolds

Rui Albuquerque

Departamento de Matemática da Universidade de Évora and Centro de Investigação em Matemática e Aplicações (CIMA), Rua Romão Ramalho, 59, 7000 Évora, Portugal.
rpa@dmat.uevora.pt

Abstract.
We use twistor theory to describe virtualy new constructions of Hermitian and quaternionic Kähler structures on tangent bundles and a G_2 structure on the unit sphere tangent bundle of a Riemannian 4-manifold — fundamental to holonomy theory and subject of deep research in physics.
We interpret "self-holomorphic" complex structures on a symplectic manifold. These complex structures give an interesting set of problems in the first possible dimension, the case of Riemann surfaces, from which should follow some interplay with Teichmuller theory, as well as with $SL(2)$ connections.

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Introduction

Twistor theory can be applied in several situations. With M. Berger's classification of non-symmetric, locally irreducible Riemannian holonomy groups in mind and looking forward for more recent studies of geometry with torsion, we were lead to the twistorial construction of old and new theories on well known bundles.

Firstly, $Sp(n)Sp(1)$ holonomy in TM where M is any $2n$-dimensional almost-Hermitian manifold. Secondly, a G_2 structure in SM, the unit sphere tangent bundle of a 4-dimensional Riemannian manifold M. The present introduction follows with a sketch of the theory behind the first case. But strongly related to the second is equation (2).

Quaternionic Kähler structures. Recall $\mathbb{H}^n$ with right action of $Sp(1)$, which is the model of a quaternionic Hermitian or $\mathbb{H}$-module. Its automorphisms constitute a subgroup $Sp(n) \subset SO(4n)$. The product subgroup $Sp(n)Sp(1)$ consists of those isometries g for which $g(vw) = g(v)w'$ for any vector v and scalar $w \in \mathbb{H}$ and some scalar w'. One can prove $Sp(n)Sp(1)$ coincides with $Sp(n) \times_{\mathbb{Z}_2} Sp(1)$.

An oriented Riemannian $4n$-manifold M is said to be quaternionic Kähler if its holonomy is inside $Sp(n)Sp(1)$ (we start with $n > 1$). We are saying M admits a quaternionic Hermitian structure, ie. there is a $\mathbb{H}$-structures on T_xM smoothly varying with $x \in M$. Such global definition means there exist locally three mutually orthogonal almost Hermitian structures I, J and $K = IJ = -JI$ generating a global endomorphisms vector bundle associated to $\mathfrak{sp}(1)$.

As it is well known from holonomy theory, the stated condition corresponds with $\nabla\mathfrak{g} \subset \mathfrak{g}$ where $\mathfrak{g}$ is (the bundle associated with) the Lie algebra of $Sp(n)Sp(1)$. As the whole structure arises from the $Sp(1)$ action, $\mathfrak{g} = \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ is closed under ∇ iff $\nabla\mathfrak{sp}(1) \subset \mathfrak{sp}(1)$ (cf. [19]).

Any two quaternionic triples I, J, K, I', J', K' on open subsets U, U', respectively, and defining the same $Sp(n)Sp(1)$-structure are related on $U \cap U'$ by a map into $SO(3)$. Denoting the 2-form $\omega_I(X,Y) = \langle IX, Y\rangle$ and ω_J, ω_K analogously, we get a well defined 4-form easily seen not to depend on the choice of I, J, K,

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K. \tag{1}$$

A straightforward computation yields, in the quaternionic Kähler case, $\mathrm{d}\Omega = 0$.

We recall that $Sp(n)Sp(1)$ is also proved to be the set of isometries of a $4n$-dimensional Euclidian fixed vector space for which a non-degenerate 4-form Ω like (1) remains invariant (V. Kraines). Thus the holonomy condition is satisfied if, and only if, $\nabla\Omega = 0$. Fortunately, it was proved in [21] that, when $n > 2$, the equation $\mathrm{d}\Omega = 0$ is also a sufficient condition for $Sp(n)Sp(1)$-holonomy. Counter-examples in dimension $n = 2$ have been found.

Hyperkähler manifolds. A Riemannian manifold is called hyperkähler if its holonomy is in $Sp(n)$. In this case we may construct a global quaternionic triple I, J, K. Now the equation for holonomy reduction $\nabla\mathfrak{sp}(n) \subset \mathfrak{sp}(n)$ implies reduction to the unitary Lie algebra or simply $\nabla I \in \mathfrak{u}(2n, I)$ — the orthogonal of $\mathfrak{sp}(n)$ in $\mathfrak{u}$ is also preserved —, which combined with $I^2 = -1$ gives $\nabla I = 0$. The same must hold for J, and hence for K. Reciprocally, from $\nabla I = \nabla J = 0$ we arrive to hyperkähler holonomy.

As it is well known, the condition is equivalent to the metric on M being Kähler with respect to each almost complex structure.

In real dimension 4. Here we have $Sp(1)Sp(1) = SO(4)$. Hence an oriented Riemannian 4-manifold M has a natural quaternionic Hermitian structure.

Since any triple I, J, K is identified to an orthonormal basis of the bundle Λ^2_+ of self-dual two forms and since $\nabla * = *\nabla$, the structure is parallel. If we select a vector field U with $\|U\| = 1$, then the $\mathbb{H}$-module structure on TM is given by

$$X_1 \cdot X_2 = (\lambda_1\lambda_2 - \langle A_1, A_2\rangle)U + \lambda_1 A_2 + \lambda_2 A_1 + A_1 \times A_2 \tag{2}$$

where $X_i = \lambda_i U + A_i$, $A_i \in U^\perp$, $i = 1, 2$, and $\langle A_1 \times A_2, A_3\rangle = \mathrm{vol}(U, A_1, A_2, A_3)$.

The name quaternionic Kähler is only given if the 4-manifold is self dual and Einstein. Such are the curvature properties of any other quaternionic Kähler metric. A hyperkähler manifold satisfies further strictness: it is self dual and Ricci flat.

TM as a Kähler or quaternionic Kähler manifold

Let M be any Riemannian manifold and D any linear metric connection on M. There exists a canonical *vertical* vector field ξ defined on the manifold TM:

$$\xi_v = v, \qquad \forall v \in TM, \tag{3}$$

under the identification of $\pi^*TM \simeq \mathscr{V} = \ker(\mathrm{d}\pi : TTM \to \pi^*TM)$, where $\pi : TM \to M$ is the canonical projection. The connection D induces a splitting $TTM = \mathscr{H}^D \oplus \mathscr{V}$.

We refer to twistor theory as used in [3, 4], after [18], where a similar canonical section ξ is defined. We use the results $X \in \mathscr{H}^D \Leftrightarrow (\pi^*D)_X\xi = 0$. Essentially, one proves that ξ varies exactly on vertical directions. Ofcourse, $\mathscr{H}^D \overset{\mathrm{d}\pi}{\simeq} \pi^*TM$.

For a given vector field $X \in \mathfrak{X}_M$ and vector $u \in T_xM$, the theory gives us a projection map $\pi^*D.\xi$ and thus $(\mathrm{d}X(u))^v = \pi^*D_{\mathrm{d}X(u)}\xi = (X^*\pi^*D)_uX^*\xi = D_uX$.

We endow TM with the canonical connection-induced metric; a metric connection D^* follows and both tools preserve the above splitting. One may write the Levi-Civita connection ∇ of TM as a function of ξ, of the curvature and of the torsion of D ([4]).

First complex structure on TM. The following map $I(X,Y) = (Y,-X)$, $X \in \mathscr{H}^D$, $Y \in \mathscr{V}$ is a compatible almost complex structure on TM. For the moment we have $D^*I = 0$.

Theorem 1 (partly [10], [4]). *(i) (TM,I) is a complex manifold iff D is torsion free and flat. If any of these occur, then M is a flat Riemannian manifold and TM is Kähler flat.*
(ii) ω_I is closed iff D is torsion free.

Notice ω_I over TM looks like the natural closed symplectic structure on the (co)-tangent bundle of any smooth manifold. These two are the same iff we consider the Levi-Civita connection of M.

Second complex structure, or a pair of them. Let $(M,\mathscr{J})$ be an almost Hermitian manifold of dim $m = 2n$. Let D denote a linear Hermitian connection: $D\mathscr{J} = 0$.

We then define two almost complex structures on TM, denoted by $J^{\pm}$: admiting again the decomposition of TTM into $\mathscr{H}^D \oplus \mathscr{V}$ we write

$$J^{\pm} = \mathscr{J} \oplus \pm\mathscr{J}. \tag{4}$$

We let, as it is usual, $T'M$ denote the $+i$-eigenbundle of $\mathscr{J}$.

Theorem 2 ([4]). *(i) J^+ is integrable iff $\mathscr{J}$ is integrable and $R^D_{u,v}\overline{w} = 0$, $\forall u,v,w \in T'M$.*
(ii) J^- is integrable iff $\mathscr{J}$ is integrable and $R^D_{u,v}w = 0, \forall u,v,w \in T'M$.
(iii) $(TM,\omega_{J^\pm})$ is symplectic iff the Hermitian connection D is flat and its torsion has no totally skew-symmetric part and is $(3,0)+(0,3)$ with respect to $\mathscr{J}$.

Third complex structure on TM. Consider the same setting as above and define J to be J^-. Consider also the previous complex structure I. Then $K = IJ = -JI$ is a new D^*-parallel almost complex structure.

Theorem 3 ([4]). *(i) K is integrable iff D is flat and torsion free iff $(M,\mathscr{J})$ is a flat Kähler manifold.*
(ii) (TM,ω_K) is symplectic if, and only if, D is torsion free. The same is to say $(M,\mathscr{J})$ is Kähler.
(iii) Suppose $n > 2$. Then the following three assertions are equivalent: (TM,I,J,K) is a quaternionic Kähler manifold; D is flat and torsion free; $(M,\mathscr{J})$ is a flat Kähler manifold. In any one of the previous cases, TM is a hyperkähler manifold.

A family of quaternionic Kähler structures on TM. Here we assume we have a $4n$ manifold M endowed with a quaternionic triple $\mathscr{J}_1, \mathscr{J}_2, \mathscr{J}_3$; Let D be the Obata connection: characterized by solving simultaneously $D\mathscr{J}_i = 0,\ i = 1,2,3$.

Now let $I_0 = I$ be the *first complex structure on TM* and let $I_i = \mathscr{J}_i \oplus -\mathscr{J}_i,\ \forall i = 1,2,3$, as the case J^- above. Notice $I_3 \neq I_1I_2 = -I_2I_1$. However, the whole four I_i anti-commute with each other. Hence for each pair $a,b \in S^3$, $a \perp b$, we have a quaternionic triple $(I_a, I_b, I_{a,b})$ given by

$$I_x = x_0I_0 + x_1I_1 + x_2I_2 + x_3I_3, \quad \forall x = a,b, \quad \text{and} \quad I_{a,b} = I_aI_b. \tag{5}$$

It is easy to verify $I_x^2 = -1$ and $I_aI_b = -I_bI_a$. A bundle with fibre the Stiefel manifold V_2^4 may thus be considered.

These structures are again quaternionic Kähler iff D is flat and torsion free. We have proved that there is a Hopf extension bundle $\{I_x\} \longrightarrow TM$ of the usual S^2-twistor bundle of a quaternionic Hermitian structure on M.

Self-holomorphic complex structures. Within the theory of bundles of complex structures, we recall the twistor space $\mathscr{Z}$ of ω-*compatible* j's over a symplectic manifold M with ω as the Kähler form. Compatible means $j \in \mathrm{End}\,TM$ with $j^2 = -1$ and for which ω is type $(1,1)$ and the metric $\omega(\ ,j\ ) \gg 0$. The study of $\mathscr{Z}$ was done in [2]. For a Riemann surface M, $\mathscr{Z}$ is a Poincaré-disk bundle and the twistor complex structure $\mathscr{J}^\nabla$ is always integrable. Recall this is induced just from a connection on M for which $\nabla\omega = 0$. So it is a $SL(2,\mathbb{R})$-connection.

We may then think of the *self-holomorphic* sections $J : M \longrightarrow \mathscr{Z}$ in any dimension:

$$\mathrm{d}J_x(JX) = \mathscr{J}^\nabla_{J(x)}\mathrm{d}J(X),\ \ \forall X \in T_xM. \tag{6}$$

This is equivalent to $\nabla_{\mathfrak{X}'}\mathfrak{X}' \subset \mathfrak{X}'$, where $\mathfrak{X}' = \Gamma(M;T'M)$. If ∇ is torsion free, then a self-holomorphic section is integrable. The Gauss curvature of the induced metric is still to be computed.

G_2 structures

Recall the exceptional Lie group $G_2 = \mathrm{Aut}\,\mathbb{O}$. The normed division algebra $\mathbb{O} = \mathbb{H}\oplus\mathbb{H}$ has product given, essentially, by mimic of the product rule of two pairs of real numbers which gives the complex line; or by mimic of the product rule of two pairs of complex numbers which gives the $\mathbb{H}$-line (such is the known Cayley-Dickson process, cf. [12]).

The imaginary part of the octonions is a subspace $\mathbb{R}^7$, where G_2 acts irreducibly. There is a well defined non-degenerate 3-form on this space: $\phi(X,Y,Z) = \langle X\cdot Y, Z\rangle$ and $G_2 = \{g \in SO(7) :\ g^*\phi = \phi\}$. These concepts all translate into a special geometry, namely that of 7-dimensional G_2-structures.

It was shown that an oriented Riemannian 7-manifold with a G_2-structure, clearly given by any non-degenerate 3-form ϕ, has holonomy in the exceptional group, apriori given by equation $\nabla\phi = 0$, if, and only if, ϕ is harmonic – a result of [11]. But less stringent G_2 classes of 7-manifolds with such a structure are given by the *torsion* forms of $d\phi$ and $\delta\phi$. These are irreducible components under the group action on Λ^4 and Λ^2 (cf. [9, 11]).

The unit sphere tangent bundle of a 4-manifold. We have found one canonical structure of the twistorial kind. Let M be an oriented Riemannian 4-manifold and let $SM = \{u \in TM : \ \|u\| = 1\} \xrightarrow{\pi} M$. let D be the Levi-Civita connection on M generating $\mathscr{H}^D \simeq \pi^*TM$. Let $\mathscr{V} = \ker d\pi$. Then $\mathscr{V}_u \perp u$ inside π^*TM since $u \in SM$ is the unique vertical direction orthogonal to $S_xM \subset T_xM$, $x = \pi(u)$. We then have an octonionic structure on

$$\mathbb{R}u \oplus T_uSM = \mathscr{H}^D \oplus \mathscr{V} \oplus \mathbb{R}u \tag{7}$$

induced by (2), with $U_u = u$, and the Cayley-Dickson process (see [3] for details).

To have a closer perception we give a hint of ϕ. Locally, we find a direct o.n. horizontal frame e_0, e_1, e_2, e_3, with e_0 corresponding to the point u. The last three vectors have natural correspondents e_4, e_5, e_6 in $\mathscr{V}$ and thence the following forms are proved to be independent of the chosen frame:

$$\mu = e^0, \quad \beta = e^{14} + e^{25} + e^{36}, \quad \alpha = e^{456}, \quad \alpha_2 = e^{126} + e^{234} + e^{315} \tag{8}$$

and $\phi = \alpha + \mu\beta - \alpha_2$.

Theorem 4 ([2])**.** *(i) SM is never a G_2 manifold:* $d\phi \neq 0$, $\forall M$.
(ii) SM is a cocalibrated G_2 manifold, ie. $\delta\phi = 0$*, if and only if M is Einstein.*

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