Gwistor spaces

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Abstract. We give a presentation of how one achieves the G_2-twistor space of an oriented Riemannian 4-manifold M. It consists of a natural $SO(3)$ structure associated to the unit tangent sphere bundle SM of the manifold. Many associated objects permit us to consider also a natural G_2 structure. We survey on the main properties of gwistor space and on recent results relating to characteristic G_2-connections with parallel torsion.

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The construction of the G_2-structure

Recall the exceptional Lie group $G_2 = \mathrm{Aut}\,\mathbb{O}$ gives birth to a special geometry. A G_2 structure on a Riemannian 7-manifold $\mathscr{S}$ is given by a stable 3-form ϕ, i.e.

- $\exists$ vector cross product $\cdot$ such that $\phi(X,Y,Z) = \langle X\cdot Y, Z\rangle$ corresponds with the octonionic product of pure imaginaries $\mathbb{R}^7 \subset \mathbb{O}$,
- $\phi = e^{456} + e^{014} + e^{025} + e^{036} - e^{126} - e^{234} - e^{315}$ on some oriented orthonormal coframe $e^0, \ldots, e^6$, or
- ϕ lives in certain open $GL(7)$-orbit of $\Lambda^3 T^*_u\mathscr{S}$.

Indeed the three statements are equivalent; the existence of such a 3-form ϕ, depending on a particular type of open orbit, implies reduction to $SO(7)$ and then reduction to G_2.

Now let T be an oriented Euclidean 4-vector space and fix $u \in S^3 \subset T$. Then T has a quaternionic structure such that $u = 1$, i.e. $\mathbb{R}u$ corresponds to the reals. The product is given by

$$(\lambda_1 u + X)(\lambda_2 u + Y) = (\lambda_1\lambda_2 - \langle X,Y\rangle)u + \lambda_2 X + \lambda_1 Y + X\times Y.$$

For the cross product $\times$ on $u^\perp$ we define $\langle X\times Y, Z\rangle = \mathrm{vol}(u,X,Y,Z)$. And conjugation in T is defined by $\overline{\lambda u + X} = \lambda u - X$.

Finally we have an octonionic structure:

$$\mathbb{O} = \mathbb{H} \oplus e\mathbb{H} \simeq \mathbb{R}u \oplus \mathbb{R}^7 \simeq T \oplus T$$

given by the Cayley-Dickson rule:

$$(z_1,z_2)\cdot(z_3,z_4) = (z_1z_3 - \overline{z_4}z_2, z_4z_1 + z_2\overline{z_3}).$$

Now let M be an oriented Riemannian 4-manifold. Let

$$SM = \{u \in TM : \ \|u\| = 1\}$$

and let $\pi : TM \longrightarrow M$ denote the tangent bundle. Since we have the classical decomposition

$$T_uSM = u^{\perp} \subset TTM = H^{\nabla} \oplus V \ \simeq^{\nabla} \ \pi^*TM \oplus \pi^*TM,$$

we may reproduce the whole construction above using the given volume form and the Sasaki metric (the pull-back of the metric on M reproduced equally in horizontal and vertical vectors and making these subspaces orthogonal). We may also consider the same construction for any generic tangent vector $u \neq 0$, taking the unit to be $1 = u/\|u\|$.

Theorem 1 *$TM\backslash 0$ admits a natural octonionic structure, i.e. there exists a vector cross product on each T_uTM, for all u, reproducing the structure of $\mathbb{O}$ and smoothly varying with $u \in TM$. The hyper-subspace SM admits a natural G_2-structure.*

It is to this last structure on SM, introduced in [5, 6], that we give the name G_2-twistor or *gwistor* space.

In order to find the 3-form ϕ of gwistor space we may proceed as follows. It is possible to construct a local orthonormal frame $e_0 = u,\ e_1, e_2, e_3 \in H^{\nabla},\ e_4, e_5, e_6 \in V$ to write the following global forms (their global characterization is given in the references):

$$\mathrm{vol} = \pi^*\mathrm{vol}_M = e^{0123}, \qquad \alpha = \text{volume 3-form on the fibres of } SM = e^{456},$$

$$\mu = e^0, \qquad \beta = e^{14} + e^{25} + e^{36},$$

$$\alpha_1 = e^{156} + e^{264} + e^{345}, \qquad \alpha_2 = e^{126} + e^{234} + e^{315},$$

$$\alpha_3 = e^{123}, \qquad \mathrm{vol} = \mu\alpha_3.$$

Finally $\phi = \alpha + \mu \wedge \beta - \alpha_2$.

A close picture to our metric structure, found in the literature, is a general contact structure which always exists on SM. Since $\mathrm{d}\mu = -\beta$ and $-\beta$ is the restriction of the natural symplectic form of TM (or the pull-back of the Liouville symplectic form through the isomorphism with the co-tangent bundle), we have a contact structure. Let $\theta^t U$ be the reflection of the canonical vertical vector field U on H^{∇}. It is the same as e_0 and, under the previous perspective, it is the so called geodesic spray of M. The contact structure was found by Y. Tashiro, 1969. In any dimension $m = n+1$, the structure

$$(SM, \frac{1}{4}g, \frac{1}{2}\mu, 2\theta^t U)$$

refers to a metric contact $2n+1$-manifold. Moreover, it is a K-contact structure if and only if $M = S^m_{\mathrm{std}}$ with sectional curvature 1.

A generalization

We may generalize the whole construction above if $H^{\nabla} \subset TTM$ comes from any given metric connection ∇ on M, i.e. instead of the Levi-Civita connection. Because all definitions are independent of the torsion.

On SM we then require the functions

$$\underline{r} = \underline{r}_u = r^\nabla(u,u), \qquad l = \circlearrowleft_{1,2,3} R^\nabla_{1230}, \qquad m = m_u = \mathrm{Tr}\, T^\nabla(u,\ ),$$

a 1-form

$$\rho_1 = r^\nabla(\ ,U) = (\mathrm{ric}\, U)^\flat,$$

the two 3-forms

$$\rho_2 = \circlearrowleft \mu(R^\nabla(\ ,\ )\ ), \qquad \sigma = \circlearrowleft \beta(T^\nabla(\ ,\ ),\ )$$

and the 4-form

$$\mathscr{R}^U \alpha := \mathrm{d}\alpha = \sum_{0\leq i<j\leq 3} R_{ij01} e^{ij56} + R_{ij02} e^{ij64} + R_{ij03} e^{ij45}.$$

Then we are able to write (we omit the wedge product)

$$\mathrm{d}\phi = \mathscr{R}^U \alpha + (\underline{r} - l)\mathrm{vol} - \beta^2 - 2\mu\alpha_1 + (\mu T)\beta - \mu\sigma - T\alpha_2,$$

$$\mathrm{d}*\phi = -\rho_2\beta - \rho_1 \mathrm{vol} - \sigma\beta - (\mu T)\alpha_1 + \mu(T\alpha_1)$$

where $T\alpha_1, T\alpha_2$ act also as derivations (cf. [2]).

Theorem 2 *We have always* $\mathrm{d}\phi \neq 0$.
For the Levi-Civita connection, $\mathrm{d}*\phi = 0$ *if and only if M is Einstein.*

For the torsion free case, ρ_2 and l vanish by Bianchi identity:

$$\mathrm{d}\phi = \mathscr{R}^U \alpha - \beta^2 - 2\mu\alpha_1 + \underline{r}\mathrm{vol}, \qquad \mathrm{d}*\phi = -\rho_1 \mathrm{vol}.$$

We also find that $(\mu T)\beta - \mu\sigma - T\alpha_2 = 0 \ \Leftrightarrow\ -\beta\sigma - (\mu T)\alpha_1 + \mu(T\alpha_1) = 0$

$$\Leftrightarrow\ T_{ijj} + T_{ikk} + T_{jkl} = 0,\ \forall\{i,j,k,l\} = \{0,1,2,3\} \text{ in direct ordering.}$$

The study of these torsion tensors (indeed frame invariant) gives the 12-dimensional solution space

$$T^\nabla \in \mathscr{A}_+ \oplus \mathscr{C}_-.$$

Recall $\Lambda^2\mathbb{R}^4 \otimes \mathbb{R}^4 = \mathbb{R}^4 \oplus \mathscr{A} \oplus \Lambda^3\mathbb{R}^4$ is the well known decomposition of the space of torsion-like tensors under the orthogonal group, due to Cartan. In the oriented case we have a further decomposition of the subspace $\mathscr{A}$ into self-dual and anti-self-dual tensors (thus under $SO(4)$). The invariant subspace $\mathscr{C}_-$ lies as a diagonal between vectorial $\mathbb{R}^4$ and totally skew-symmetric torsion, also 4-dimensional. We are referring to the subspace

$$\mathscr{C}_\pm = \left\{T:\ T(X,Y,Z) = \nu(X)\langle Y,Z\rangle - \nu(Y)\langle X,Z\rangle \pm 2\nu^\sharp \lrcorner \mathrm{vol}_M(X,Y,Z),\ \nu \in \mathbb{R}^{4^*}\right\}$$

The torsion forms

Recall that, by Fernandez-Gray decomposition of the subspaces $\Lambda^p\mathbb{R}^7$ as G_2-modules, $0 \leq p \leq 7$, there must exist global differential forms $\tau_i \in \Lambda^i$ such that

$$\mathrm{d}\phi = \tau_0 * \phi + \frac{3}{4}\tau_1\phi + *\tau_3, \qquad \mathrm{d}*\phi = \tau_1 * \phi + *\tau_2.$$

In our choice of orientation, the invariant subspace of τ_2 is the space of 2-forms satisfying $\tau_2\phi = *\tau_2$ and τ_3 satisfies $\tau_3\phi = \tau_3 * \phi = 0$, cf. [2, 8]. For SM with $T^\nabla = 0$ we deduce

$$\tau_0 = \frac{2}{7}(\underline{r}+6), \quad \tau_1 = -\frac{1}{3}\rho_1, \quad \tau_2 = \frac{1}{3}\rho_1^\sharp\lrcorner(\phi - 3\alpha),$$
$$\tau_3 = *(\mathscr{R}^U\alpha) - \frac{2}{7}(\underline{r}-1)\phi + (\underline{r}-2)\alpha + \frac{1}{4}*(\rho_1\phi).$$

The reader finds formulas for these so called torsion forms, in the general case of a metric connection with torsion, in reference [2].

Example: if $M = \mathscr{H}^4$ is locally a real hyperbolic space with sectional curvature -2, then SM is of pure type W_3:

$$\mathrm{d}\phi = *\tau_3 = *(2\mu\beta - 6\alpha), \qquad \mathrm{d}*\phi = 0.$$

Examples with 4-dimensional rank 1 symmetric spaces M yield homogeneous gwistor spaces, i.e. the action on M lifts to a transitive action on SM:

$$S\mathbb{S}^4 = V_{5,2}, \qquad S\mathscr{H}^4 = \frac{SO_0(4,1)}{SO(3)}, \qquad S\mathbb{CP}^2 = SU(3)/U(1) = N_{1,1}$$

Since any tangent space $T_{\mathbb{C}z}\mathbb{CP}^2 = \mathbb{C}^3/\mathbb{C}z$, the subgroup $U(1) \subset SU(3)$ corresponds with

$$\begin{bmatrix} e^{it} & & \\ & e^{it} & \\ & & e^{-2it} \end{bmatrix}.$$

Non of the natural G_2-twistor structures in the cases above correspond with other known in the literature on the same spaces. We do find an original homogeneous example with the G_2-twistor space of hyperbolic Hermitian space

$$\mathscr{H}^2_{\mathbb{C}} = \frac{SU(2,1)}{S(U(2)\times U(1))}.$$

Characteristic connection

According to some approach to string theory through G_2 geometry, cf. [1, 10, 11], it is important to find the metric connections on the space which preserve the structure. In particular the characteristic connection:

$$\nabla^{\mathrm{c}} = \nabla^g + \frac{1}{2}T^{\mathrm{c}},$$

$$\nabla^c g = 0, \qquad T^c(X,Y,Z) = \langle T^{\nabla^c}(X,Y), Z\rangle \in \Lambda^3 \qquad \text{and} \qquad \nabla^c \phi = 0.$$

Applying the theory to find the characteristic connection of (SM, ϕ), we get:

- If M is Einstein, SM has a unique characteristic G_2-connection.
- If M has constant sectional curvature k, then $T^c = (2k-2)\alpha - k\mu\beta$.

We deduce after some computations:

Theorem 3 ([4]) *The gwistor space SM has parallel characteristic torsion,* $\nabla^c T^c = 0$*, if and only if* $k = 0$ *or* 1.

So now we study the two cases separately. If $k = 1$ we have locally the Stiefel manifold $V_{5,2} = SS^4$, with the gwistor structure.

Theorem 4 ([4]) *(i) If* $k = 1$*, then the characteristic* G_2*-connection* $\nabla^c = \nabla^g - \frac{1}{2}\mu\beta$ *of* $V_{5,2}$ *coincides with the characteristic contact connection of the Sasakian manifold* $S\mathbb{S}^n = SO(n+1)/SO(n-1) = V_{n+1,2}$ *in case* $n = 4$.

Moreover ∇^c *agrees with the invariant canonical connection of the homogeneous space. Hence it is complete and with holonomy* $SO(n-1)$.

(ii) If $k = 0$*, then the characteristic* G_2*-connection* $\nabla^c = \nabla^g - \alpha$ *of* $\mathbb{R}^4 \times S^3$ *is flat.*

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